Math 310: **Final Exam** (Solutions)
Prof. S. Smith: Friday 13 Dec 2002

All 5 problems are worth 20 points each. You must SHOW WORK to receive credit.
(If you use a calculator, WRITE "I used calculator" at those places).
For Problem 1 (part (a) on page 1, and part (b) on the reverse side page 2), write your work ON THE CORRESPONDING SIDE OF THIS SHEET.
For Problems 2–5 on page 3, write you work IN YOUR EXAM BOOK.

Problem 1: (subspaces and linear transformations)

(a) Let S denote the set of vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in $\mathbf{R}^2$ satisfying the condition $2x_1 - x_2 = 0$.
Show that S is a subSPACE of $\mathbf{R}^2$, via the following steps:

(closure under addition:)

(1) Write down a general vector in S (that is, name its coordinates). Answer: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(1+) What does it mean that your vector is in S ? (Its coordinates satisfy what condition ?)
Answer: $2x_1 - x_2 = 0$. *Or, you could say the vector has the form* $\begin{pmatrix} a \\ 2a \end{pmatrix}$

(2) Write down another general vector in S (name its coordinates): Answer: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

(2+) What does it mean that your new vector is in S ? (... condition ?)
Answer: $2y_1 - y_2 = 0$.

(3) Now add your two vectors from (1) and (2). Answer: $\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$

(3+) What do you need, for the sum vector in (3) to be in S ? (condition on its coordinates ?)
Answer: $2(x_1 + y_1) - (x_2 + y_2)$ *must equal* 0.

(3++) Now verify the condition in (3+) (show how to use (1+) and (2+)).
Answer: *The left-hand side in (3+) re-arranges by the distributive law etc into:*
$(2x_1 - x_2) + (2y_1 - y_2)$; *and each term is zero by (1+) and (2+).*

(closure under scalar multiplication:)

(4) Write down a general vector in S (you can use (1) again). Answer: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(4+) What does it mean that your vector is in S ? (Its coordinates satisfy what condition ?)
Answer: $2x_1 - x_2 = 0$.

(5) Write down a general scalar in $\mathbf{R}$ (name it): Answer: c

(6) Now multiply the vector in (4) by the scalar in (5). Answer: $\begin{pmatrix} c\,x_1 \\ c\,x_2 \end{pmatrix}$

(6+) What do you need, for the product vector in (6) to be in S ? (condition ...?)
Answer: $2(c\,x_1) - (c\,x_2)$ *must equal* 0.

(6++) Now verify the condition in (6+) (show how to use (4+)).
Answer: *The left-hand side in (6+) re-arranges by the distributive law etc into:*
$c\,(2x_1 - x_2)$; *and then the right factor is zero by (4+).*

Part (b) of Problem 1 is on the reverse side of this page.

(b) Let L denote the function from $\mathbf{R}^2$ to $\mathbf{R}^2$ defined by $L(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} 2x_2 \\ 3x_1 + x_2 \end{pmatrix}$.

Show that L is a LINEAR transformation, via the following steps:

(to show same answer for addition, before or after L:)

(1) Write down a general vector (that is, name its coordinates). Answer: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(2) Write down another general vector (name its coordinates): Answer: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

(3) Now add your two vectors from (1) and (2). Answer: $\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$

(3+) Now apply L to the sum vector in (3): Answer: $\begin{pmatrix} 2(x_2 + y_2) \\ 3(x_1 + y_1) + (x_2 + y_2) \end{pmatrix}$

(4) Now apply L separately to the two vectors in (1) and (2).

Answers: $\begin{pmatrix} 2x_2 \\ 3x_1 + x_2 \end{pmatrix}$ and $\begin{pmatrix} 2y_2 \\ 3y_1 + y_2 \end{pmatrix}$

(4+) Now add the two vectors in (4). Answer: $\begin{pmatrix} 2x_2 + 2y_2 \\ (3x_1 + x_2) + (3y_1 + y_2) \end{pmatrix}$

FINALLY: Are the answers in (3+) and (4+) the same?

Answer: *Yes, using distributive laws ...*

(to show same answer for scalar multiplication, before or after L:)

(5) Write down a general vector (you can use (1) again). Answer: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(6) Write down a general scalar (name it): Answer: c

(7) Now form the product of the vector in (5) and the scalar in (6). Answer: $\begin{pmatrix} c\ x_1 \\ c\ x_2 \end{pmatrix}$

(7+) Now apply L to the product vector in (7): Answer: $\begin{pmatrix} 2(c\ x_2) \\ 3(c\ x_1) + (c\ x_2) \end{pmatrix}$

(8) Now apply L to the vector in (5). Answer: $\begin{pmatrix} 2x_2 \\ 3x_1 + x_2 \end{pmatrix}$.

(8+) Now multiply the vector in (8) by the scalar in (6). Answer: $\begin{pmatrix} c\ (2x_2) \\ c\ (3x_1 + x_2) \end{pmatrix}$

FINALLY: Are the answers in (7+) and (8+) the same?

Answer: *Yes, using distributive laws ...*

The remaining problems (2)–(5) are on the next page.

Problem 2: Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(a) Find the characteristic polynomial, and the eigenvalues, of A.

$\det(A - xI) = (2-x)(2-x) - 1 = (x^2 - 4x + 4) - 1 = x^2 - 4x + 3 = (x-3)(x-1)$, *so eigenvalues are* $1, 3$.

(b) Find the eigenspaces for those eigenvalues.

For 1: $A - 1.I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ *has rref* $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, *get solutions* $a(-1,1)^T$.

For 3: $A - 3.I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, *has rref* $\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$, *get solutions* $b(1,1)^T$.

Problem 3: Given the differential equation system (functions of t): $\begin{pmatrix} y_1' & = & 6y_1 & -4y_2 \\ y_2' & = & 3y_1 & -y_2 \end{pmatrix}$.

I GIVE you the information that eigenvalues of the coefficient matrix A for this system are $2, 3$,

(a) Find eigenvectors for these eigenvalues of A; then use them to give the *general* solution of the system (with undetermined constants c_1, c_2).

For 2, get $a(1,1)^T$*; for 3, get* $b(\frac{4}{3}, 1)^T$.

Then solution vector $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix} e^{3t}$ *so* $y_1 = c_1 e^{2t} + \frac{4}{3} c_2 e^{3t}$ *and* $y_2 = c_1 e^{2t} + c_2 e^{3t}$.

(b) Now find the particular solution (values of c_1, c_2) given initial values $y_1(0) = 6$, $y_2(0) = 5$.

Solve $\left(\begin{array}{cc|c} 1 & \frac{4}{3} & 6 \\ 1 & 1 & 5 \end{array}\right)$ *to get* $c_1 = 2, c_2 = 3$. *So* $y_1 = 2e^{2t} + 4e^{3t}$ *and* $y_2 = 2e^{2t} + 3e^{3t}$.

Problem 4: (a) For $A = \begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix}$, I GIVE you that eigenvalues are $2, 4$.

Find eigenvectors for these eigenvalues; and give a matrix X with $X^{-1}AX$ a diagonal matrix D. (Show X^{-1} and D also).

For 2, get $a(-1,1)^T$*; for 4, get* $b(-1,3)^T$.

We can use $X = \begin{pmatrix} -1 & -1 \\ 1 & 3 \end{pmatrix}$, $X^{-1} = -\frac{1}{2} \begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix}$ *with* $D = X^{-1}AX = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$.

(b) Let $A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$. GIVEN: the eigenvalues of A are $1, 2, 2$.

Find the DIMENSIONS of the eigenspaces for these eigenvalues.

Is A diagonalizable? Say why/why not.

Check that $rref(A - 1.I_3)$ *has 1 free variable, so the dimension of the 1-eigenspace is 1. Similarly* $rref(A - 2.I_3)$ *has 1 free variable, so the dimension of the 4-eigenspace is 1.*

Then A is not diagonalizable—since for the eigenvalue 2, the dimension of the eigenspace is less than the number of times the eigenvalue appears as a root of the characteristic polynomial. (That is, geometric multiplicity $<$ *algebraic multiplicity for 2).*

Problem 5: Let A be the symmetric matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

I GIVE you the eigenvalues $2, -1, -1$ of A; and an eigenvector $(1,1,1)^T$ for eigenvalue 2.

(a) Find a basis of the eigenspace of A for eigenvalue -1.

The row-reduced echelon form of $A - (-1).I = A + I$ *has* $(1,1,1)$ *as its only nonzero row.*
So eigenvectors are $(-b-c, b, c)^T$*; and one possible basis is* $(-1,1,0)^T$ *and* $(-1,0,1)^T$.

(b) Now find an *orthonormal* basis for the eigenspace in (a).
Use it to give an orthogonal diagonalization of A;
that is, find an *orthogonal* matrix X (satisfying $X^{-1} = X^T$) with $X^{-1}AX$ diagonal.

For 2: eigenspace is 1-dimensional; divide original $(1,1,1)$ *by its length* $\sqrt{3}$: $x_3 = \frac{1}{\sqrt{3}}(1,1,1)^T$.
For 1: Start with above basis like $v_1 = (-1,1,0)^T$ *and* $v_2 = (-1,0,1)^T$.
Apply Gram-Schmidt: first $q_1 = (-1,1,0)$,
and then $q_2 = v_2 - \frac{v_2 \cdot q_1}{q_1 \cdot q_1} q_1 = (-1,0,1)^T - \frac{1}{2}(-1,1,0)^T = (-\frac{1}{2}, -\frac{1}{2}, 1)^T$
so may as well use the more convenient multiple $q_2 = (1,1,-2)^T$.
Now divide each by its length, to get $x_1 = \frac{1}{\sqrt{2}}(-1,1,0)^T$ *and* $x_2 = \frac{1}{\sqrt{6}}(1,1,-2)^T$.

So now putting x_3 *first, can use* $X = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$.