Math 310: **Final Exam** (Solutions)

Prof. S. Smith: Tues 7 Dec 1999

You must SHOW WORK to receive credit. (If you use a calculator, INDICATE those places where you use it).

Problem 0 (review) is worth 10 points, and Problems 1–5 are worth 20 points. So the maximum score possible is 110.

Problem 0: (As promised, a flashback to Hour Exam 2:)

Let V be the space $\mathbf{F}^{2\times 2}$ of 2×2 matrices. Let I be the usual 2×2 identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Determine the orthogonal complement, that is, the subspace $I^\perp$ of all 2×2 matrices orthogonal to I. (Recall that the inner product of two matrices is defined by $\langle A, B\rangle = \sum_{i,j=1}^{2} A_{ij}B_{ij}$).

A is in $I^\perp$ if $0 = \langle A, I\rangle = A_{11}.1 + A_{12}.0 + A_{21}.0 + A_{22}.1 = A_{11} + A_{22}$.

Thus $I^\perp$ consists of the matrices of form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$.

Problem 1: Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. (a) Find the eigenvalues of A.

$\det(A - xI) = (x-1)^2 - 4 = x^2 - 2x - 3 = (x-3)(x+1)$, *so eigenvalues are* $3, -1$.

(b) Find the eigenspaces for those eigenvalues.

For 3: $A - 3.I = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$, *via* $A_2^{-1\times 1}$ *to* $\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$, *get solutions* $a(1,1)^T$.

For -1: $A - (-1).I = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$, *via* $A_2^{-1\times 1}$ *to* $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$, *get solutions* $b(-1,1)^T$.

Problem 2: Given the differential equation system (functions of t): $\begin{pmatrix} y_1' & = & y_1 & +y_2 \\ y_2' & = & -2y_1 & +4y_2 \end{pmatrix}$.

I GIVE you the information that eigenvalues of the coefficient matrix A for this system are $2, 3$, with corresponding eigenvectors $(1,1)^T$ and $(1,2)^T$.

(a) Give the *general* solution of the system (with undetermined constants c_1, c_2).

$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$ *so* $y_1 = c_1 e^{2t} + c_2 e^{3t}$ *and* $y_2 = c_1 e^{2t} + 2c_2 e^{3t}$.

(b) Now determine the particular solution (values of c_1, c_2) for the initial value problem $y_1(0) = 2$, $y_2(0) = 1$.

Solve $\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 2 & 1 \end{array}\right)$ *to get* $c_1 = 3, c_2 = -1$.

So $y_1 = 3e^{2t} - e^{3t}$ *and* $y_2 = 3e^{2t} - 2e^{3t}$.

Problem 3: (a) Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

I GIVE you that the eigenvalues of A are $0, 1, 1$. Is A diagonalizable? Indicate why/why not.

Yes: we compute that the eigenspace of 0 is $a(-1,0,1)^T$, and the eigenspace for 1 is $(0,b,c)$ of dimension 2. Thus we CAN find a basis of eigenvectors; alternatively, the geometric multiplicity equals the algebraic multiplicity for each eigenvalue.

(b) For $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, I GIVE you that eigenvalues are $3, 1$. Diagonalize A: this is, find X with $X^{-1}AX$ diagonal. Use this to determine the exponential e^A.

We compute that eigenvectors are $(1,1)^T$ *and* $(1,-1)^T$.

So we can use $X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $X^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ *with* $D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

So $A = XDX^{-1}$ *and then* $e^A = X(e^D)X^{-1}$ *where* $e^D = \begin{pmatrix} e^3 & 0 \\ 0 & e \end{pmatrix}$.

Multiplying out we get $e^A = \frac{1}{2}\begin{pmatrix} e^3 + e & e^3 - e \\ e^3 - e & e^3 + e \end{pmatrix}$.

Problem 4: Let A be the symmetric matrix $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$.

I GIVE you that the eigenvalues of A are $-2, -2, 1$.

(a) Find a basis for each eigenspace of A.

For 1: eigenvectors are $a(1,1,1)^T$.

For -2: Get eigenvectors $(-b-c, b, c)^T$. *So one possible basis is* $(-1,1,0)^T$ *and* $(-1,0,1)^T$.

(b) Give an orthogonal diagonalization of A; that find an orthogonal matrix X (satisfying $X^{-1} = X^T$) with $X^{-1}AX$ is diagonal.

For 1: eigenspace is 1-dimensional; divide previous vector by its length: $\frac{1}{\sqrt{3}}(1,1,1)^T$.

For -2: Start with above basis like $(-1,1,0)^T$ *and* $(-1,0,1)^T$.

Apply Gram-Schmidt to get $\frac{1}{\sqrt{2}}(-1,1,0)^T$ *and* $\frac{1}{\sqrt{6}}(1,1,-2)^T$. *So can use* $X = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$.

Problem 5: (a) For the Markov matrix $A = \begin{pmatrix} .7 & .2 \\ .3 & .8 \end{pmatrix}$, find the "steady-state" vector v. (That is, $Av = v$, and the coordinates of v add up to 1.)

The eigenspace for 1 *consists of vectors* $a(2,3)^T$. *So the steady-state vector is* $(.4, .6)^T$.

(b) Is the symmetric matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ positive definite? (Why/why not?)

Yes: e.g., the determinants of the principal minors are positive: $1, 1, 1$. *(Alternatively, the eigenvalues are positive; but the computer is needed to find them...)*