Math 310: **Notes on Sec 3.3 Exercise 8 (HW week 6)**
Prof. S. Smith: Mon 1 Mar 1999

The calculations for Exercise 8 of Section 3.3 can be complicated; these notes will summarize several ways of working the problem.

You're asked whether the three "vectors" $\cos x$, 1, $\sin^2(\frac{x}{2})$ are linearly independent in the function space $C[-\pi, \pi]$.

(Easiest) By eye: If your trigonometric identities are not too rusty, you can use the standard "half-angle" formula

$$\sin^2(\frac{x}{2}) = \frac{1 - \cos x}{2}$$

to see that the third is a linear combination of the first two, and so the functions are LD. Equivalently, use the visible numbers as coefficients in a non-zero linear combination giving 0:

$$-\frac{1}{2}(\cos x) + \frac{1}{2}(1) - 1(\sin^2(\frac{2}{2})) = 0$$

(Messier) Wronskian method: Set up the 3 functions and 1st and 2nd derivatives in a determinant:

$$\det \begin{pmatrix} \cos x & 1 & \sin^2(\frac{x}{2}) \\ -\sin x & 0 & 2\sin(\frac{x}{2})\cos(\frac{x}{2})(\frac{1}{2}) \\ -\cos x & 0 & [-\sin^2(\frac{x}{2}) + \cos^2(\frac{x}{2})](\frac{1}{2}) \end{pmatrix}$$

In class today (1 Mar 99) I started in on the computation and got bogged down. Instead, the thing to do is to first simplify using related trigonometric identities for the two lower-right terms, namely

$$\sin\frac{x}{2}\cos\frac{x}{2} = \frac{1}{2}\sin x \text{ and } \cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2}) = \cos x$$

so that the determinant (by Laplace expansion down 2nd column) is $-\sin x(\frac{1}{2}\cos x) + \cos x(\frac{1}{2}\sin x) = 0$; that is, all terms do drop out, so that the Wronskian is just 0. And therefore the 3 functions are LD.

Evaluation method: Here we set up a linear combination, and set it equal to zero. But instead of leaving it as a combination of functions, we evaluate the function at some different places. If the resulting specialization gives us equations that have only the trivial all-zeros solution, then we know the functions are LI.

However, if this method fails, it does NOT guarantee that the functions are LD—we might have chosen the points of evaluation in some special way. So the method is not really appropriate for this problem, where the answer is actually LD, but the method is not powerful enough to prove that fact.

That is, if in this problem we picked certain points (say $-\pi, 0, \pi$) then the equations would have an infinite number of solutions. But that would only give a combination for the values at those points—not for all points—so it wouldn't be a complete proof that the 3 functions are LD.