COHOMOLOGY DECOMPOSITIONS
FROM SUBGROUP COMPLEXES OF FINITE GROUPS

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ABSTRACT. A survey of parts of the research area of "subgroup complexes"—which combines techniques from finite group theory, algebraic topology, and combinatorics.

The exposition emphasizes the historical development, from the particular viewpoint of applications to *simple* groups and their geometries. The foundations are provided by the work of Tits on buildings, and of Brown and Quillen on simplicial complexes of p-subgroups of a group. The discussion continues through some relatively recent theoretical contributions from homotopy theory and from the p-local structure of simple groups; these improvements have made possible a joint work of Benson and Smith, which provides for each of the 26 sporadic simple groups a "small, simplex" decomposition of its mod 2 group cohomology.

This is a more extended written version of a keynote talk delivered at the Midlands regional meeting of the London Mathematical Society, at the University of Birmingham on 18 May 2005—in conjunction with the subsequent workshop on "Fusion systems, representation theory, and groups".

CONTENTS

1991 *Mathematics Subject Classification.* Primary: 20J06; Secondary: 20J05, 55R35, 55P91.
Key words and phrases. cohomology decomposition, subgroup complex, sporadic group.
The author was partially supported by NSA grant MDA 904-03-1-0100.

INTRODUCTION

The goal of this talk is to survey part of the research area of "subgroup complexes": namely the material providing the natural background for a joint research project with Dave Benson [BS][1] over the last several years.

So I will begin with a rough statement summarizing our main result; planning that the subsequent exposition will serve to define any unexplained terms in the statement:

Theorem 0.1 (Benson–Smith [BS]). *Each of the* 26 *sporadic simple groups* G *admits a "small, simplex" decomposition of its mod-*2 *cohomology; where the indexing simplex is the orbit complex for the action of* G *on a suitable* 2*-local geometry* Δ.

In addition, the exposition to follow should indicate how some comparatively recent technological advances in the area of subgroup complexes has made possible the proof of the result in the final cases, corresponding to the largest sporadic groups.

1. HISTORICAL CONTEXT OF SIMPLE GROUPS AND SUBGROUP COMPLEXES

With the above preview in place, let's now pass back to the earlier history of geometries for simple groups, and simplicial complexes defined by chains of subgroups.

The "types" of finite simple groups. In the statement of Theorem 0.1, only the first author should be credited with any actual expertise in the cohomology of groups! Instead I have worked mainly in finite group theory, especially in the classification of the simple groups, and in trying to understand their structure and representation theory.

Let's recall that the Classification of the Finite Simple Groups[2] customarily describes the (nonabelian) simple groups in terms of the following three types:

- alternating groups $A_n (n \geq 5)$;
- groups of Lie type; and
- 26 sporadic groups (not fitting into any infinite family such as those above).

The Lie type groups really do deserve to be considered as a single type: They are well studied and understood via a uniform description, as suitable analogues of the infinite matrix groups over $\mathbb{R}$ and $\mathbb{C}$ of the Lie theory—but with matrix entries taken from a finite field, of some prime characteristic p. A useful example to keep in mind

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[1] Our writeup runs about 300 pages; so it will probably be submitted to a monograph series such as *Memoirs of the A.M.S.* Much of that length is in fact devoted to an extensive exposition of the background material from algebraic topology and from group theory—since we suspected that most readers would be expert in only one of those areas. We are now polishing up that exposition, and would expect to web-post a preprint version soon.

[2] I'll take this opportunity to spread the word: As the reader may by now be aware, the widely lamented gap in the proof of the Classification was recently closed, with the publication in November 2004 of the treatment of quasithin groups by Aschbacher and Smith in [AS04a, AS04b]. However, the work described here does not quote the Classification; it only uses facts about the structure of the known simple groups. So any reader still uneasy with the Classification can just prefix the word "known", any time that the phrase "simple groups" occurs in this talk.

is the full linear group $GL_n(\mathbb{F}_p)$ of all $n \times n$ invertible matrices over the field of order p.[3]

By contrast, the 26 sporadic groups are individual exceptions, which arise in various ways, and are not so well understood; they are not really a "type", and are only grouped together in order to avoid listing 26 further separate cases. Yet it is possible to hope for some more uniform understanding of the sporadic groups–by looking for at least partial analogies with the Lie type groups.

So for this talk, we will be regarding the Lie type groups as the *model case* for the simple groups.

Geometries for simple groups. One classical way of trying to understand groups is to examine their actions on suitable geometrical structures. For today's purposes, a *geometry* for a group G will be just some simplicial complex Δ with a G-action.

Given a geometry Δ, we can then study various aspects of the topology of Δ, such as its homology groups. In particular today we will encounter the *reduced*[4] *Lefschetz module* $\tilde{L}(\Delta)$, which is the classical alternating sum of the homology groups; this formal sum gives a *virtual* RG-module for any choice of coefficient ring R. In fact by $\tilde{L}(\Delta)$ we will usually mean the variant provided by the alternating sum of the chain groups—which by the Hopf trace formula has the same character. The chain groups can in turn be broken up into sums of permutation modules $R[G/G_\sigma]$, where G_σ denotes the subgroup of G stabilizing a representative σ of some G-orbit on simplices of Δ.

As an example of a geometry, note that the alternating group A_n acts naturally on the n vertices of a simplex Δ of dimension $n-1$. Next consider the linear group $GL_n(\mathbb{F}_p)$, regarded as the group of a vector space V of dimension n over $\mathbb{F}_p$. We obtain a geometry from the usual *projective space* of V—as the *order complex* of that poset (partially ordered set) under inclusion: that is, we consider the *flags* in V, namely inclusion-chains of nonzero proper subspaces of V; since a nonempty subflag of a flag is again a flag, the flags form a simplicial complex Δ.

Tits in [Tit74] generalized vastly from this classical example of projective space for the linear group: he showed that *every* group G of Lie type has an analogous natural geometry Δ called its *building*. The building is constructed as a simplicial complex of inclusion-chains in the poset of subgroups of G given by the *parabolic subgroups*—the analogues in G of the subgroups of the linear groups stabilizing flags of subspaces. We mention also that if G is defined in characteristic p, then each parabolic has a normal p-subgroup called its *unipotent radical*; and the building can also be viewed as inclusion-chains of unipotent radicals. The geometry of Δ is again uniformly described via the Lie theory, in terms of the Weyl group of G and its Dynkin diagram.

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[3] For our expository purposes, it is "close enough" to being simple: to be precise, one should consider only matrices of determinant 1, and take the quotient modulo scalar matrices. But these details have little effect on the overall structural properties that the group exemplifies.

[4] of course this means that we include a formal term in dimension -1

The theory of buildings has been enormously influential. As just one example relevant for today's purposes, we mention the *Solomon-Tits theorem*: which states that for G of Lie type in characteristic p, the reduced Lefschetz module $\tilde{L}(\Delta)$ for the building using coefficient ring $R := \mathbb{F}_p$ in fact just reproduces[5] the celebrated *Steinberg module* of G: namely the unique irreducible $\mathbb{F}_pG$-module which is also *projective* in characteristic p.

This is of course one type of geometric result that we might wish to imitate for other groups. The pioneering work in the application of the geometric viewpoint to other simple groups, and especially to sporadic groups, was that of Buekenhout in [Bue79], focusing on axiom systems corresponding to diagrams closely resembling the Dynkin diagrams used for Lie type groups. The geometric approach attracted the interest of many group researchers; and since the early 1980s, "groups and geometries" has been a vigorous research area in group theory—in which the Benson–Smith project mentioned above has followed just one particular direction.

For today's purposes, we will focus in particular on the 2-*local geometries* for sporadic groups G—since they seem to be especially well suited for consideration of the group cohomology of G. The 2-local geometries were introduced by Ronan and Smith in [RS80], from the viewpoint of treating suitable 2-local subgroups (namely the normalizers of nontrivial 2-subgroups) of a sporadic group G as suitable analogues of the parabolic subgroups of a group of Lie type in characteristic 2.[6]

Complexes from posets of p-subgroups. The study of simplicial complexes obtained from posets of subgroups of a group G also became a vigorous research area in algebraic topology[7] during the 1970s. Here the focus was not on a simple group G; instead G could be *any* finite group—or a compact Lie group, or certain other directions of generalization. But today we will state things as restricted just to *finite* groups.

From now on, let p denote some fixed prime dividing the order $|G|$ of our group G. In particular, *we take the coefficient ring R to be* $\mathbb{F}_p$, when we consider the reduced Lefschetz module $\tilde{L}(\Delta)$ of some complex Δ for G.

Brown in [Bro75] introduced the study of the poset $\mathcal{S}_p(G)$ of all nontrivial p-subgroups of G; this determines the order complex denoted $|\mathcal{S}_p(G)|$, of inclusion-chains of members of $\mathcal{S}_p(G)$. The result which perhaps attracted the greatest attention is the following congruence on the *Euler characteristic* χ of the complex (which in its reduced form gives the degree term of $\tilde{L}(\Delta)$):

$$\chi(\ |\mathcal{S}_p(G)|\ )\ \equiv\ 1 \pmod{|G|_p}\ ;$$

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[5] In this case, the alternating sum just reduces to the top homology group, since one can show that the other homology groups vanish here

[6] The same ideas are also developed in the literature for odd primes p. However, there are fewer interesting examples: for the odd primes p tend to divide the order of the sporadic groups to a fairly low power, and the p-local structure is then correspondingly less rich. The other main cases studied usually have $p = 3$ or 5.

[7] Also combinatorialists such as Stanley, Garsia, and Björner were developing the related area now usually called *poset combinatorics*.

this can be regarded as a homological analogue of the statement in Sylow's theorem that the number of Sylow p-subgroups of G is congruent to 1 mod p.

Subsequently Quillen in a very influential paper [Qui78] developed the homotopy properties of such p-subgroup complexes in a number of important directions.

One emphasis of Quillen was the possibility of passing from $\mathcal{S}_p(G)$ (which might be very complicated if $|G|_p$ is large) to a smaller and more convenient subposet: if the subposet is *homotopy equivalent* to $\mathcal{S}_p(G)$, then all homological information is preserved. Quillen focused particularly on the subposet $\mathcal{A}_p(G)$ of all nontrivial *elementary abelian* p-subgroups. It is also relevant to mention here that a little later, Bouc introduced the subposet $\mathcal{B}_p(G)$ of nontrivial *p-radical* p-subgroups—that is, P satisfying $P = O_p\big(N_G(P)\big)$, where O_p denotes the largest normal p-subgroup of a group. We summarize [Qui78, 2.1] and [Bou84, Cor., p. 50] as:

(1.1) The inclusion of $\mathcal{A}_p(G)$ or $\mathcal{B}_p(G)$ in $\mathcal{S}_p(G)$ induces a homotopy equivalence.

Thus we see that the equivalence class of $\mathcal{S}_p(G)$ under homotopy equivalence, namely the *homotopy type* of $\mathcal{S}_p(G)$, contains both $\mathcal{A}_p(G)$ and $\mathcal{B}_p(G)$. (Later we will need to consider some related, but distinct, homotopy types.) Quillen also made a particularly dramatic application of this viewpoint to the case of Lie type groups, showing that this homotopy type also contains the Tits building:

Proposition 1.2 (3.1 in [Qui78])**.** *For G of Lie type in characteristic p, the building Δ of G is homotopy equivalent to $\mathcal{S}_p(G)$.*

In fact Quillen showed equivalence with $\mathcal{A}_p(G)$, so here we are implicitly also using (1.1); and another now-standard proof proceeds instead via $\mathcal{B}_p(G)$, checking that these subgroups are exactly the unipotent radicals of the parabolic subgroups, and so afford that version of the building. (Thévenaz and Webb observed later in [TW91] that all these equivalences can be strengthened to G-homotopy equivalences, so these posets in fact lie in the G-homotopy type of $\mathcal{S}_p(G)$.)

Another direction emphasized by Quillen was the projectivity of the reduced Lefschetz module for these posets; we will state the result in a strengthened form due to Webb—see later (2.2):

(1.3) For Δ in the homotopy type of $\mathcal{S}_p(G)$, $\tilde{L}(\Delta)$ is a virtual $\mathbb{F}_p$-projective.

In view of 1.2, we see that the projective property of the Steinberg module of a group of Lie type can be viewed as a special case of (1.3). Consequently $\tilde{L}(\ |\mathcal{S}_p(G)|\ )$ is sometimes regarded as an analogue of the Steinberg module for an arbitrary finite group G. However, we emphasize that $\tilde{L}(\ |\mathcal{S}_p(G)|\ )$ does not in general exhibit the *irreducible* property that the Steinberg module has when G is of Lie type.

In due course many group theorists also became interested in this area of poset topology; particularly influential in this regard was Webb's expository lecture at the 1986 AMS Summer Symposium in Arcata, and its published form [Web87b]. Webb emphasized projectivity, with applications to decompositions of group cohomology;

and in particular he examined some geometries for sporadic groups. So for the remainder of this talk, we will now specialize the general context considered so far to this particular topic of cohomology decompositions—especially from the viewpoint of the Benson–Smith project on all the sporadic groups.

2. COHOMOLOGY DECOMPOSITIONS FOR GENERAL FINITE GROUPS

We continue the context above, where G can be any finite group, and p is a prime dividing its order.

Webb considered some general simplicial complex Δ with a G-action; and focused on the following technical property:

Hypothesis 2.1. *Assume, for all nontrivial p-subgroups P of G, that the fixed subcomplex Δ^P is contractible.*

The (elementary) verification of 2.1 for $\mathcal{S}_p(G)$ was a main technical ingredient in the proofs of Brown and Quillen of the results indicated above, for the case where Δ is in the homotopy type of $\mathcal{S}_p(G)$. Correspondingly Webb extended various results to more general Δ; the following analogue of (1.3) appears as [Web87a, Thm A']:

(2.2) For Δ satisfying Hypothesis 2.1, $\tilde{L}(\Delta)$ is a virtual $\mathbb{F}_p$-projective.

Webb now emphasized the consequences of this result for group cohomology; from now on, *by H^* we will mean cohomology taken with coefficients in $\mathbf{F}_p$.* Since the Ext-functors defining cohomology vanish on projectives, Webb deduced the following decomposition of group cohomology over the quotient CW-complex Δ/G, in terms of the cohomology of the subgroups G_σ stabilizing simplices:

Proposition 2.3 (Theorem A in [Web87a])**.** *If $\tilde{L}(\Delta)$ is a virtual $\mathbb{F}_p$-projective, then:*

$$H^*(G) = \bigoplus_{\sigma \in \Delta/G} H^*(G_\sigma) \tag{2.4}$$

In particular this holds if Δ satisfies Hypothesis 2.1.

The formula (2.4) is often called *Webb's formula*; it is what we mean by a *cohomology decomposition* over Δ, for example in the statement of 0.1 at the start of the talk.

We caution that such decompositions describe only additive structure; and do not necessarily provide information about the multiplicative structure of the cohomology ring. Nonetheless, additive decompositions have often proved useful in the computation of cohomology for interesting groups; for example, in the program of Adem and Milgram and their collaborators, in which they have been describing the cohomology rings of more and more of the sporadic groups—starting with [AMM91] which treats the sporadic Mathieu group M_{12}.

Notice 2.3 shows that the existence of *some* cohomology decomposition for G is not surprising—since Hypothesis 2.1 holds for Δ in the homotopy type of $\mathcal{S}_p(G)$. However those particular decompositions are not necessarily optimal:

- The poset $\mathcal{S}_p(G)$ can be very *large*, when p divides $|G|$ to a high power.

- The quotient structure $|\mathcal{S}_p(G)|/G$ can be a very *complicated* CW-complex.

Alleviating these difficulties will lead to the desirable "small" and "simplex" properties for a decomposition which we mentioned when stating 0.1.

We mention two fairly obvious methods for trying to remedy the first difficulty above, by passing to a simplicial complex Δ smaller than $|\mathcal{S}_p(G)|$. One approach is

▶ **Method (i):** Show Δ is homotopy equivalent to $|\mathcal{S}_p(G)|$,

for example as in (1.1). But we will see that even $\mathcal{A}_p(G)$ and $\mathcal{B}_p(G)$ need not be minimal for the purpose of decompositions. Another approach is

▶ **Method (ii):** Directly check Hypothesis 2.1 for Δ.

Furthermore we might also hope to remedy both difficulties if we pass to a more specific class of groups G, where we can exploit structure which is not available for all finite G—for example, *simple* groups. We turn first to the model case given by Lie type groups, where we can apply Method (i) via 1.2:

3. THE MODEL DECOMPOSITION: LIE TYPE GROUPS AND BUILDINGS

Let G denote a group of Lie type in characteristic p, and Δ its building.

We can proceed via Method (i) above using homotopy equivalence: By 1.2, we obtain the projectivity result (1.3) for Δ: and hence by 2.3:

(3.1) We obtain the Webb decomposition formula (2.4) over the building.

This decomposition[8] has two very special properties. First, it is

- "small", in the sense of minimal dimension:

in fact, Δ has dimension given by one less than the Lie rank of G; and one can show that this dimension is suitably minimal for the purpose of decompositions of cohomology. Second, it is a

- "simplex" decomposition, in the sense that the quotient Δ/G is a simplex.

Namely the action of G on its building Δ is *flag-transitive*, so that the quotient Δ/G is just a single simplex, of the dimension indicated just above—rather than some more difficult CW-complex. This simplex property produces a significant simplification in the indexing of the sum in the decomposition: namely the G-orbits of simplices σ can be indexed by proper subsets J, of a set I indexing the classes of maximal parabolic subgroups; and then (3.1) takes the form

$$H^*(G) = \bigoplus_{\emptyset \neq J \subseteq I} H^*(G_J) \qquad (3.2)$$

These "small, simplex" aspects of the decomposition for Lie type groups provided specific motivation for our search for analogous decompositions for sporadic groups G—culminating in our recent completion of the final cases for Theorem 0.1. Indeed

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[8] In spite of the availablity of this wonderful decomposition, it remains an open problem in general to describe the mod p cohomology of a Lie type group in characteristic p. Furthermore only scattered facts are known about the cohomology of those groups with mod q coefficients for q other than the characteristic p.

a main reason for choosing the 2-local geometries Δ for G described earlier is that they similarly exhibit "simplex" decompositions—since G is also flag-transitive on Δ in those cases;[9] and so the decompositions in 0.1 take the form (3.2). Furthermore the decompositions are also "small"—indeed in for a number of sporadic groups G, the 2-local geometry Δ has dimension strictly less than that of the order complexes of the standard general posets such as $\mathcal{A}_2(G)$ or $\mathcal{B}_2(G)$.

Thus for the proof of Theorem 0.1, the important point is to show that the decomposition over Δ really *does* compute the group cohomology $H^*(G)$. We will proceed via Methods (i) and (ii) indicated above, or via suitable variations on them.

4. Sporadic simple groups described by the homotopy type of $\mathcal{S}_2(G)$

The proof of Theorem 0.1 for 11 of the 26 sporadic groups, namely

$$M_{11}, M_{22}, M_{23}, M_{24}, J_1, J_3, J_4, McL, Co_2, Th, Ly,$$

in fact follows via Method (ii) from work done during the 1980s in groups and geometries: Namely Ryba, Smith, and Yoshiara in [RSY90] determined which of the then-known sporadic geometries satisfy Hypothesis 2.1. For $p = 2$, that condition turned out to hold for the 2-local geometries Δ for the first 10 of the sporadic groups listed above. The appropriate geometry for the 11th case, namely the sporadic group Ly of Lyons, was not discovered until later; and Hypothesis 2.1 for it is in fact verified in the present Benson–Smith work [BS].

Hence (2.2) gives the projectivity of $\tilde{L}(\Delta)$; and that (rather than cohomology) was in fact the primary interest of the authors in [RSY90]. But of course that projectivity also gives the hypothesis of 2.3, and hence the cohomology decompositions there over Δ. This observation by Benson was essentially the beginning of our project in [BS]; in particular it establishes the indicated 11 cases of Theorem 0.1.

However in the light of later developments, it is probably more appropriate to mention that those 11 cases can also be approached via Method (i) of homotopy equivalences: Smith and Yoshiara observed in [SY97] that for most p-local geometries, Hypothesis 2.1 in fact leads to a homotopy equivalence of Δ with $\mathcal{S}_p(G)$. Thus for $p = 2$, these 11 sporadic groups are the ones whose 2-local geometry Δ lies in the homotopy type of $\mathcal{S}_2(G)$; so that the decomposition over Δ can also be deduced using (1.3) rather than (2.2) to provide the hypothesis of 2.3.

But notice that this remark also indicates the difficulty that remained in order to establish Theorem 0.1 for the other 15 sporadic groups: For Method (ii) turned out to be just Method (i) for $\mathcal{S}_p(G)$, and so we have run out of methods. In other words, the projectivity hypothesis in 2.3 turned out to be essentially equivalent to the homotopy type of $\mathcal{S}_p(G)$; so it was necessary to have further methods for decompositions, which apply to homotopy types *other* than the standard one for $\mathcal{S}_p(G)$—and then one can get more mileage out of Method (i) for those further homotopy types.

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[9] In fact, in four of the sporadic groups, namely J_3, McL, Fi'_{24}, and Ly, because of certain identifications we can take the decomposition in 0.1 even over a suitable subset of the simplex.

5. Decompositions of the classifying space BG from homotopy theory

Webb's formula for general Δ in (2.4) was also one of the motivations for a more general study of decompositions—not just at the algebraic level of the cohomology groups of G, but also at the underlying topological level of the *classifying space* BG of G; such decompositions of BG have been developed in recent years by homotopy theorists, for example in the study of compact Lie groups.

(We recall that the classifying space BG is a topological space, designed so that its cohomology as a space just reproduces the group cohomology of G. In one standard construction of BG due to Milnor, the classical bar resolution for the cohomology of G is modeled by taking G as a discrete space, and forming the infinite join EG of that space—then finally obtaining $BG = EG/G$ as the quotient modulo the free action arising from that of G on itself by translation.)

A number of directions in the area of decompositions were subsequently analyzed by Dwyer in a unified setting; we recommend particularly his recent exposition [Dwy01], as a further reference for some of the material sketched below.

The Borel construction for a complex and the normalizer decomposition. Before describing some of the other decompositions from homotopy theory, we first mention Dwyer's analysis in the previous case of (2.4)—since that "normalizer decomposition" is the particular form we use in the more extended statement of Theorem 0.1 given below at 5.1.

We saw above that Webb's original approach in [Web87a] to the decomposition (2.4) used Hypothesis 2.1 to obtain the projectivity assumption for 2.3. In a later approach in [Web91], Webb used Hypothesis 2.1 instead to provide a sufficient condition for "approximating" the classifying space BG by the *Borel construction* $(EG \times \Delta)/G$—an approximation in the sense that these two spaces should have the same mod-p cohomology.

In Dwyer's subsequent analysis, that Borel construction is in fact regarded as $\underrightarrow{\mathrm{Hocolim}}_{\sigma \in \Delta/G} BG_\sigma$—using the *homotopy colimit* construction of Bousfield and Kan [BK72] to assemble together the component classifying spaces for the subgroups G_σ. Since we may regard G_σ as the normalizer $N_G(\sigma)$, the corresponding homotopy colimit is called the *normalizer decomposition*. When it gives an isomorphism in mod p cohomology with that of BG, the decomposition is called *ample*. When the *spectral sequence* associated to the homotopy colimit collapses, the decomposition is called *sharp* (which in fact includes being ample). In this case of the normalizer decomposition, this collapse leads to the particularly simple alternating-sum form of Webb's formula in (2.4).

Next we proceed one step beyond the analysis of Dwyer: The *p-completion* of a space is another construction of Bousfield and Kan in [BK72], which in particular takes a mod-p cohomology isomorphism, and produces a homotopy equivalence at the level of completions.

So using this language, we can now add to our earlier statement of Theorem 0.1 some further content at the level of the underlying topological spaces:

Theorem 5.1 (Benson–Smith [BS]). *For the* 26 *sporadic groups* G*, and the corresponding* 2*-local geometries* Δ *in Theorem 0.1, the cohomology decomposition indicated there in fact follows from these properties:*

(i) The normalizer decomposition $\underset{\sigma\in\Delta/G}{\underrightarrow{\mathrm{Hocolim}}}\; BG_\sigma$ *is ample and sharp.*

(ii) The 2*-completion* $(\underset{\sigma\in\Delta/G}{\underrightarrow{\mathrm{Hocolim}}}\; BG_\sigma)^\wedge_2$ *of that homotopy colimit is homotopy equivalent to the* 2*-completed classifying space* $BG^\wedge_2$ *of* G*.*

The usefulness of (ii) is that a standard further property of completions says for any prime p that the p-completed classifying space is a space which affords exactly the mod-p cohomology of G.

Now probably many group theorists and other potential readers of [BS] (not to mention the general audience for this talk!) will not have the detailed background in homotopy theory needed for such constructions as the homotopy colimit and the p-completion—as well as the context of simplicial sets in place of topological spaces which they require. It is for this reason that we are providing in the Benson–Smith work [BS] a fairly extensive exposition (on the order of 200 pages) of this background material—preparatory to the detailed arguments on the individual sporadic groups which actually establish our main results.

Further decompositions and further p-subgroup collections. We can now move on to describing two other decompositions from homotopy theory. Though in each case we will start with $\mathcal{A}_p(G)$ or $\mathcal{B}_p(G)$, which we saw at (1.1) are in the homotopy type of $\mathcal{S}_p(G)$—we will see that the further methods may apply to suitable subposets, which can then determine other homotopy types.

We first consider the decomposition in the work of Jackowski and McClure in [JM92], which is based on $\mathcal{A}_p(G)$—we emphasize that rather than considering just the simplicial complex determined by this poset with its inclusions, they use a homotopy colimit which is indexed instead by a more complicated category, involving as morphisms not just inclusions but also G-conjugations. In this viewpoint the relevant stabilizers are not normalizers but centralizers, and so they use the *centralizer decomposition* $\underset{E\in\mathcal{A}_p(G)/G}{\underrightarrow{\mathrm{Hocolim}}}\; BC_G(E)$—which they show is ample and sharp.

In analyzing the hypotheses actually used in their proof, Jackowski and McClure also showed that the same holds for a subposet which we will here call $\mathcal{E}_p(G)$. This poset may be proper, and can determine a homotopy type distinct from that of $\mathcal{A}_p(G)$. For example, this situation often happens when there is some conjugacy class, consisting of elements of order p central in some Sylow p-subgroup, which is closed under taking nontrivial commuting products.

We mention that because of the more complicated indexing category, the sharpness of the centralizer decomposition does not translate into a formula as simple as Webb's alternating sum in (2.4) for the normalizer decomposition. However, the normalizer decomposition turns out to still be available, in view of the following piece of more recent technology; namely among the results of Grodal and Smith in [GS][10] is:

(5.2) The normalizer decomposition for $\mathcal{E}_p(G)$ is ample and sharp.

These methods can in fact be applied to sporadic groups: Benson in [Ben94] observed that the relevant condition for a proper subposet $\mathcal{E}_2(G)$ holds for the case of the Conway group Co_3. This observation was in effect the beginning of our work in establishing Theorem 0.1 for at least some of the remaining 15 sporadic groups which do not lead to the homotopy type of $\mathcal{S}_2(G)$. We will return to this point, after introducing one more decomposition from homotopy theory:

In contrast with the above discussion of the centralizer decomposition related to $\mathcal{A}_p(G)$, the work of Jackowski, McClure, and Oliver in [JMO92a, JMO92b] is based on $\mathcal{B}_p(G)$. They also consider a more complicated category, with objects given by G-orbits of form G/H for $H \in \mathcal{B}_p(G)$, and natural morphisms corresponding suitably to inclusions and conjugations. In this viewpoint, the relevant stabilizers are not normalizers but just subgroups, such as H for the orbit G/H; and they use the *subgroup decomposition* $\underset{H \in \mathcal{B}_p(G)/G}{\underrightarrow{\mathrm{Hocolim}}}\ BH$—which they show is ample and sharp.

Later Dwyer in [Dwy97, 8.10] established the same conclusions for the subposet $\mathcal{B}_p^{cen}(G)$ of $\mathcal{B}_p(G)$ of members P which are *p-centric*; that is P contains all p-elements of $C_G(P)$. Often this subposet is proper, and determines a different homotopy type than that of $\mathcal{B}_p(G)$. Again because of the more complicated category, the sharpness of the subgroup decomposition does not translate into a simple alternating sum as for the normalizer decomposition. But again, the latter is still available by virtue of more recent technology; among the results of Grodal in [Gro02][11] is:

(5.3) The normalizer decomposition for $\mathcal{B}_p^{cen}(G)$ is ample and sharp.

We mention also that Grodal obtains similar results for a further subposet called the *principal* p-radical subgroups $\mathcal{D}_p(G)$. However for the sporadic groups we consider, this does not usually seem to differ from $\mathcal{B}_p^{cen}(G)$.

6. SPORADIC SIMPLE GROUPS DESCRIBED BY OTHER HOMOTOPY TYPES

We have now established the background which allows us to describe the process of establishing Theorem 0.1 for the remaining 15 sporadic groups: In summary, we can now use Method (i) to make use of the above technology from homotopy theory, for homotopy types other than that of $\mathcal{S}_2(G)$—namely we show that the 2-local geometry

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[10] The Grodal–Smith paper actually determines in a comprehensive way the interrelations of sharpness properties among the decompositions from all the standard p-subgroup collections. However, the general analysis originated when the second author pestered the first for a proof of this particular result (5.2) needed for [BS].

[11] Grodal's paper contains a very illuminating general analysis of decompositions, supplementing that of Dwyer. But again the particular result (5.3) here was at least partly stimulated by the guess in [SY97] that this result should hold.

Δ is homotopy equivalent either to $\mathcal{E}_2(G)$ or to $\mathcal{B}_2^{cen}(G)$. The equivalence methods used are essentially just those in [SY97].

However, that process also requires the use of some other rather recent advances—this time from group theory: For working with the posets $\mathcal{E}_2(G)$ and $\mathcal{B}_2^{cen}(G)$ presupposes adequate knowledge of $\mathcal{A}_2(G)$ and $\mathcal{B}_2(G)$; and this information (at least for the larger sporadic groups) was not available until recently. We mention in particular the work of Meierfrankenfeld and Shpectorov [MS] determining the 2-local subgroups of the Monster M and the Baby Monster B; this work was in turn used by Yoshiara in [Yos] to complete the determination for all sporadics G of the posets $\mathcal{B}_p(G)$ (earlier work of Yoshiara and a number of other authors, notably J. An, had treated the smaller sporadic groups).

It turns out that 6 of the 26 sporadic groups, namely

$$M_{12}, J_2, HS, Suz, Co_3, Ru,$$

have a 2-local geometry Δ which can be shown to be homotopy equivalent to $\mathcal{E}_2(G)$. Thus for these groups, we use (5.2) to obtain the relevant cases of Theorem 0.1.

Similarly 11 of the 26 sporadic groups, namely

$$Suz, Co_1, Fi_{22}, Fi_{23}, Fi'_{24}, HN, B, M, He, Ru, O'N,$$

have a 2-local geometry which can be shown to be homotopy equivalent to $\mathcal{B}_2^{cen}(G)$. Thus for these groups, we use (5.3) to obtain the relevant cases Theorem 0.1.

The alert reader will have noticed that in fact two of the sporadic groups, namely Suz and Ru, fall into the intersection of the two above categories: that is, for them the homotopy types of $\mathcal{E}_2(G)$ and $\mathcal{B}_2^{cen}(G)$ agree.

7. Some possible future directions

Here are some contributions to one possible future avenue of research—which is partly suggested by a closer look at the results in [BS]; I plan to indicate more of the suggestive details in a short talk in the workshop here later this week:

We saw earlier that for 15 sporadic groups G, the 2-local geometry Δ is not in the homotopy type of $\mathcal{S}_2(G)$; and this corresponds with $\tilde{L}(\Delta)$ not being projective. So one can ask, what *is* the nature of the module $\tilde{L}(\Delta)$?

For example: in those 15 cases, the degree $\tilde{\chi}(\Delta)$ of the module is not divisible by the full 2-power $|G|_2$; but it is still divisible by 2-power "almost" that high. Indeed it was observed long ago, using a viewpoint due to Thévenaz, that the module $\tilde{L}(\Delta)$ is *relatively* projective with respect to suitable 2-subgroups of small order.

This approach is further developed via the 2-modular representation theory of G, in forthcoming work of Sawabe [Saw]: Sawabe shows that such small 2-subgroups are given, in some of the sporadic cases discussed above with respect to $\mathcal{B}_2^{cen}(G)$, by the collection of small subgroups which are 2-radical but not 2-centric; and these 2-subgroups are distinguished by the property that no larger subgroup can appear as a *vertex* of an indecomposable direct summand of $\tilde{L}(\Delta)$. In particular this gives an "explanation" of the 2-power appearing in the degree $\tilde{\chi}(\Delta)$.

In fact, I believe that a more specific aspect of the 2-modular representation theory, namely block theory, should provide a still deeper explanation of this phenomenon:

In essence, the ampleness of Δ in 5.1 is saying that the module $\tilde{L}(\Delta)$ is *acylic* in the sense of having zero mod p reduced cohomology. For the 11 sporadic groups with Δ in the homotopy type of $\mathcal{S}_2(G)$, this is certainly explained by the projectivity of $\tilde{L}(\Delta)$ in (1.3). However, it can also be explained via the modular representation theory of G, *if all non-projective components of* $\tilde{L}(\Delta)$ *lie in blocks other than the principal block*; that is, in a block other than that containing the trivial character, for which the *defect group* is the full Sylow group. Now Benson and Wilkerson in [BW95] had made just such an observation for the case of M_{12}: namely that $\tilde{L}(\Delta)$ lies entirely in a non-principal block; in fact, in a block of defect 2 (as compared to defect 6 for the principal block). To begin to follow up this indication, I had a quick look at all 15 of the sporadic groups with Δ not in the homotopy type of $\mathcal{S}_2(G)$: in each case, one can check at least the *existence* of a non-principal block of small positive defect—which would exactly explain the 2-power in $\tilde{\chi}(\Delta)$. In particular each such small defect group should then appear as a vertex of an indecomposable summand of $\tilde{L}(\Delta)$, giving a deeper underlying explanation of the maximal vertices in the above analysis of Sawabe. However all of this remains conjectural—since the modules $\tilde{L}(\Delta)$ have been explicitly computed only for the smaller sporadic groups. I do hope to investigate this more in the future (and in particular to get Dave Benson in on the analysis). But maybe others also would be interested in looking at this phenomenon.

Finally I mention a somewhat different direction, which is also partly suggested by the analysis of the various cases in [BS]: In order to treat a number of the 15 sporadic groups with Δ not in the homotopy type of $\mathcal{S}_2(G)$, we chose to indicate the homotopy equivalence of Δ with one of the standard 2-subgroup posets (say $\mathcal{E}_2(G)$), because the other natural candidate poset (say $\mathcal{B}_2^{cen}(G)$) seemed more difficult to approach—and possibly even might lie in a different homotopy type. But I am pleased to report that in at least some of these cases, John Maginnis and Silvia Onofrei are giving a methodical treatment of the details of the homotopy equivalence relationships among the 2-local geometries and the various relatives of $\mathcal{A}_2(G)$ and $\mathcal{B}_2(G)$; I have seen preliminary versions of their work on Co_3, and on McL in [MO].

So I'll close by just reiterating my conviction that the combination of topological techniques with those from algebra is strongly advancing our understanding of the sporadic simple groups.

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