CFSG—A User's Manual: Applying the Classification of Finite Simple Groups

(longer version, handout format; as of 28sept2015)

Venice Summer School
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Topics of the lectures:

1: An introduction to the groups in the CFSG lec1 lec1a
2: An outline of the proof of the CFSG lec2 lec2a
3: (Failure of) Thompson factorization—etc lec3 lec3a
4: Recognition theorems for simple groups lec4 lec4a
5: Representation theory of simple groups lec5 lec5a
6: Maximal subgroups of simple groups lec6 lec6a
7: Geometries for simple groups lec7 lec7a
8: Some fusion techniques for classification problems lec8 lec8a
9: Some more group-theoretic applications lec9 lec9a
10: Some applications farther afield lec10 lec10a

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

A note on some references

(Later I'll indicate some other useful sources, as they arise.)

I particularly thank Sergei Gelfand/AMS for permission
to provide during-the-course (taken down, later Sept 2015)
online versions of some of my own (joint) publications:

As one reference on the CFSG, I often use the "outline":

- Aschbacher–Lyons–Smith–Solomon [ALSS11].

Some other sources used occasionally will include:

- (quasithin) Aschbacher-Smith [AS04a, AS04c];
- (Quillen Conjecture) Aschbacher-Smith [AS93];
- (Subgroup Complexes) Smith [Smi11, 0.0.1].

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Lecture 1: An introduction to the groups in the Classification of Finite Simple Groups

(version of 27sept2015)

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Venice Summer School
7-18 September 2015

Overview of the talk:

Introduction: Statement of the CFSG
1: Alternating groups
2: Sporadic groups
3: Groups of Lie type

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

Foreword (to the whole course)

What is the purpose of classifying simple groups?
(As with any other type of mathematical object:)
Primarily for *applications*:
in the many problems which reduce to simple groups.

To that end, we (first) need the *list* of simple groups;
but also, various *properties* of the groups in the list.

Comments:
(1) The CFSG proof uses induction on order:
get simple composition factors of proper(!) subgroups;[1]
so, the CFSG proof is also an "application" of the CFSG–list.
(2) The list-groups and their properties are also often applied to problems, eg classifications, which need not quote CFSG.

That said, let's get started—with the fundamental CFSG-list:

---

[1] called a $\mathcal{K}$-*group*—for the list $\mathcal{K}$ of "known" simple groups

Introduction: Statement of the CFSG

The classification is usually stated in this (summary) form:

CFSG Theorem: (E.g., [ALSS11, Thm 0.1.1])

A (nonabelian)[2] finite simple group G is one of:

i) An alternating group $A_n (n \geq 5)$.

ii) A group of Lie type ("most" simple groups are here).

iii) One of 26 *sporadic* groups.

These summary "types" don't even include the group names...

So we expand below—along with some of their properties.

[2] Warning: Often when I write "simple", I implicitly mean "nonabelian simple". Ideally this intention will always be clear from the context.

As a main source for simple groups and their properties,
I mainly use "GLS3": Gorenstein-Lyons-Solomon [GLS98].
(But often also: Wilson's excellent book [Wil09].)

What are some kinds of properties of interest?

- subgroups (maximal; p-local; elem ab; p-rank $m_p(-)$; ...)
- extensions (automorphisms $\mathrm{Out}(G)$; Schur multiplier; ...)
- representations (permutation; linear: char $0 +$ char p; ...)

Today we'll start some of these. Many more, in later lectures.

So now let's explore the various types of simple groups:

§1: Alternating groups

Here A_n is just the even permutations, inside symmetric S_n.
Regarded as familiar; and elementary to work with...
(So we often use A_n—or even S_n—to explore examples.)

Properties? First a few standard observations:

- multiply transitive: S_n is n-trans (and A_n is $(n-2)$-trans);
- indeed S_n is transitive on partitions of any fixed type.

More detailed properties are given in Sec 5.2 of [GLS98];
E.g. p-rank (at 5.2.10.a): $m_p(A_n) = \lfloor \frac{n}{p} \rfloor$ for odd p.
(Partition n via size-p parts. **Exer 1.1:** For $p = 2$?)

Maximal subgroups? These are a little subtle (more in lec6):

- "obvious" cases: via *structures*, eg from partitioning n;
  Ex: $A_k \times A_{n-k}$
- but simple groups can also arise—not thus predicted...

(See O'Nan-Scott Theorem: lec6; e.g. [Wil09, Thm 2.4].)
Exer 1.2: Give some structures in small cases—A_5, A_6, ...

§2: Sporadic groups

Basically means: not in any natural *infinite* family.
But the list of 26 is often subdivided into short "families"; e.g.:

Mathieu groups: M_{11}, M_{12}, M_{22}, M_{23}, M_{24}.

- Arise as multiply-transitive subgroups of corresp A_n.
- Also: acting on Steiner systems, and related Golay codes.

Conway groups: Co_3, Co_2, Co_1.

- Arise from automorphisms of 24-dim Leech lattice.

Fischer groups: Fi_{22}, Fi_{23}, Fi_{24}'.

- Arise as groups generated by 3-transpositions (later lec4).
- Have 2-locals with 2-group extended by M_{22}, M_{23}, M_{24}.

Janko groups: J_1, J_2, J_3, J_4.

- Discovered by Janko; but in separate contexts...

Monster series: *Har*, *Th*, *BM* (Baby Monster), *M* (Monster)

- found by Harada, Thompson, Fischer, Fischer-Griess
- arise from automorphisms of 196884-dim Griess algebra. (denoted F_5, F_3, F_2, F_1 in Atlas [C^+85])

The rest: There are 7 further; named for their discoverers:

$$He,\ HS,\ Ly,\ McL,\ O'N,\ Ru,\ Suz$$

(Held, D. G. Higman and Sims, Lyons,
McLaughlin, O'Nan, Rudvalis, and Suzuki.)

Various of these have (limited) inter-relations; e.g.:

- some arise as rank-3 permutation groups (cf. later lec4);
- some appear in the Leech-lattice context; etc...

But seemingly no really cogent "family" patterns.

For properties of each, see Sec 5.3 of [GLS98];
or Atlas [C^+85], or Griess [Gri98], or Aschbacher [Asc94] ...

§3: Groups of Lie type

Since "most" simple groups are of Lie type,
we'll now spend more time on this case.
(Beginning group theorists should definitely study this area...)

I learned ($\sim$1973) using Carter's book [Car89];
for the algebraic-groups approach, cf. [Car93].
Much information is collected in Chapters 1–4 of [GLS98].

Prior to sketching the fully general context,
let's explore some features in a more concrete case:
the linear group GL_n—to allow "pictures" (i.e. matrices).
(OK, not quite simple—but close-enough, for exposition...)

Standard Example: $G := GL_n(q)$; $q = p^a$ for a prime p.
Realize as matrices on natural module V of dim n over $\mathbb{F}_q$.

First, one basic feature (cf. partition-transitivity of S_n):
Transitive on decomps $V = V_1 \oplus V_2 \oplus \cdots$ via fixed dims;
and similarly on flags $V_1 < V_2 < \cdots$ of fixed type.

Next for certain *sub*groups of G, we can draw pictures:
For structure relevant to p, consider $B := UH$, where:

$$U := \begin{pmatrix} 1 & & \\ * & 1 & \\ * & * & 1 \end{pmatrix},\ H := \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

Then lower-triangular U is a p-group; indeed a Sylow group;
as is upper-tri U^-; note $\langle U, U^- \rangle = SL_n(q)$—most of G.
Further $N_G(U) = UH = B$, for H the diagonal matrices.
(And $N_G(H) = HW$, for W the permutation matrices.)

A more general p-local, the k-space stabilizer $P_k = U_k L_k$:

$$U_k := \left(\begin{array}{c|c} I_k & 0 \\ \hline * & I_{n-k} \end{array}\right),\ L_k := \left(\begin{array}{c|c} GL_k(q) & 0 \\ \hline 0 & GL_{n-k}(q) \end{array}\right)$$

Here U_k is also a p-group; with $N_G(U_k) = U_k L_k = P_k$.

These preview notions from the more general Lie theory:
—unipotent, semisimple elements;
—Borel, Cartan, Weyl, monomial, parabolic subgroups; ...

Other matrix groups: We can also draw matrix pictures, in the other *classical* (matrix) groups;
i.e., groups preserving a form on a space V:

- orthogonal (symmetric form);
- symplectic (anti-symmetric);
- unitary (conjugate-symm: w.r.t. field aut of order 2).

A traditional ref is Artin [Art88b]; more recent, Taylor [Tay92].

One crucial property is **Witt's Lemma**; e.g. [GLS98, 2.7.1];
this roughly gives transitivity on decompositions and forms
(so again analogous to multiple transitivity of S_n).

Exer 1.3: Mimic the linear example above, in small cases...
Hint: For *hyperbolic* basis in the form,
take "opposite" pairs $\langle v_1, v_n \rangle \perp \langle v_2, v_{n-2} \rangle \perp \cdots$
This gives some symmetry about the "anti"-diagonal ($\diagup$);
e.g. Sylow U is again lower-triangular ...

A sketch of the more general theory of Lie type groups

We'll first need a page or two of (classic) background:

Some theory of simple Lie algebras $\mathcal{G}$ over $\mathbb{C}$.

One standard reference is Humphreys [Hum78];
for the relation to Lie groups/$\mathbb{C}$, I like Varadarajan [Var84].

The structure of such a $\mathcal{G}$ is described via a standard setup:

- a *root system* $\Phi = \Phi^+ \cup \Phi^-$
  (with a basis—a *simple* system $\Pi \subset \Phi^+$),
- a *Weyl group* W acting on Φ,
- all axiomatized via a *Dynkin diagram*.

There are corresponding nilpotent subalgebras $\mathcal{U}^{\pm}$,
and a Cartan subalgebra $\mathcal{H}$.

Cartan-Killing classified such $\mathcal{G}$ (e.g. [Hum78, 11.4])
in fact by classifying their root systems—
—with the corresponding Dynkin diagrams given by:

Dynkin diagrams (Lie *types*, for simple Lie algebras / $\mathbb{C}$)

$$
\begin{array}{lll}
\text{(classical)} & A_n\colon & \circ - \circ - \circ - \;\cdots\; - \circ - \circ \\
 & B_n, C_n\colon & \circ - \circ - \circ - \;\cdots\; - \circ = \circ \\
 & D_n\colon & \circ - \circ - \circ - \;\cdots\; - \circ <^{\circ}_{\circ} \\
\text{(exceptional)} & E_6\colon & \circ - \circ - \overset{\overset{\circ}{|}}{\circ} - \circ - \circ \\
 & E_7\colon & \circ - \circ - \overset{\overset{\circ}{|}}{\circ} - \circ - \circ - \circ \\
 & E_8\colon & \circ - \circ - \overset{\overset{\circ}{|}}{\circ} - \circ - \circ - \circ - \circ \\
 & F_4\colon & \circ - \circ = \circ - \circ \\
 & G_2\colon & \circ \equiv \circ
\end{array}
$$

Also note graph symmetries—will lead to *twisted*-type groups:

Reflection: $\leftrightarrow$, to get ${}^2A_n$, ${}^2B_2$, ${}^2E_6$, ${}^2F_4$, ${}^2G_2$; $\updownarrow$ for ${}^2D_n$.

Rotation $\curvearrowright$ of order 3: to get ${}^3D_4$.

From algebras to groups: the Chevalley construction

Over $\mathbb{C}$: can exponentiate nilpotent $\mathcal{U}^{\pm}$ to unipotent $U^{\pm}$,
to generate Lie group G over $\mathbb{C}$. But over finite $\mathbb{F}_q$?
Chevalley adjusted the exp-process—e.g. [Car89, Ch 4];
to get corresponding Lie *type* group $G(q)$ over $\mathbb{F}_q$:

- find suitable (Kostant) $\mathbb{Z}$-form $\mathcal{G}_{\mathbb{Z}}$;
- *can* exponentiate nilp $\mathcal{U}^{\pm}_{\mathbb{Z}}$—to unip $U^{\pm}$; generating $G(q)$.

These finite $G(q)$ inherit much theory from Lie groups;
and most (i.e. except some small cases) are simple.

Further: Can obtain the (infinite) *algebraic* group $\overline{G}$,
by working instead over algebraic closure $\overline{\mathbb{F}_p}$.
Then $G(q)$ (where $q = p^a$) also arise, via fixed-points $\overline{G}^{\sigma}$:
where σ is a-th power of Frobenius aut $x \mapsto x^p$.
And get further groups, of *twisted* types, as $\overline{G}^{\sigma\tau}$,
for τ a suitable graph symmetry.

It remains to relate these Lie types to more explicit viewpoints:

The Lie types—with some other naming conventions

Lie type X	usual term for $X(q)$	usual notation
(classical matrix:)	(incl some twisted)	
A_n	linear	L_{n+1}
${}^2A_n$	unitary	U_{n+1}
B_n	orthogonal	Ω_{2n+1}
C_n	symplectic	Sp_{2n}
D_n	orthogonal	Ω^+_{2n}
${}^2D_n$	orthogonal	Ω^-_{2n}
(exceptional:)		
G_2, F_4, E_6, E_7, E_8		
(non-cl twisted:)	(some say "excep")	
${}^2B_2(2^{\mathrm{odd}})$	Suzuki	$Sz(2^{\mathrm{odd}})$
${}^3D_4$	triality D_4	
${}^2F_4(2^{\mathrm{odd}})$	(due to Ree)	
${}^2E_6$	twisted E_6	
${}^2G_2(3^a)$	Ree	$Ree(3^a)$

Some features of p-locals in our GL_n example
re-appear in the theory of *parabolics* (e.g. [Car89, Sec 8.3]):

The ("1-parameter") *root subgroups* U_α for $\alpha \in \Phi^+$ generate
a *full unipotent* group U (Sylow); with $\langle U, U^- \rangle = G$.
Next from $[U_\alpha, U_{-\alpha}]$ we can extract a "diagonal" group H_α;
these generate a *Cartan* (diagonal) subgroup H.
We have $N_G(U) = UH =: B$, a *Borel* subgroup; and
monomial gp $N := N_G(H)$ with $N/H \cong W$. (So: *BN-pair*.)

Overgroups of B (and conjugates) are parabolic subgroups:
A parabolic P_J is determined by a subset $J \subseteq \Pi$; $P_J = U_J L_J$,
with *unipotent radical* U_J generated by U_α, $\alpha \in \Phi^+ \setminus \Phi_J$;
and *Levi complement* L_J generated by H and U_α, $\alpha \in \Phi_J^\pm$.
We have $N_G(U_J) = P_J$; and further $N_G(P_J) = P_J$.
Note: $\#(\text{maxl locals} \geq \text{Sylow } U) = \#(\text{nodes in diagr}) = |\Pi|$
Borel-Tits Thm [GLS98, 3.1.3]: any p-local is in a parabolic.

Exer 1.4: Express earlier classical cases P_k (etc) via J, Π.

We'll PAUSE here; before moving on to applications...

Lecture 1a: Some easy uses of the CFSG-list
(version of 25sept2015)

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Overview of the talk:
Theme: Using the CFSG-list of simple groups, to describe:
1: Components—in the generalized Fitting subgroup
2: Outer automorphisms of simple groups (Schreier Conj)
Afterword: Determining multiply-transitive groups

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

Some references for applications of the CFSG

Let me first indicate several surveys,
which contain a wealth of material on applications:

- (permutation groups) Cameron [Cam81];
- (representation theory) Tiep [Tie14];
- (maximal subgroups) Kleidman-Liebeck [KL88];
- (various areas) Kantor [Kan85];
- (many areas) Guralnick (to appear in Proc ICM-2014)

I have used various topics from these sources;
and I strongly recommend them to the interested reader.

(Intro:) How does knowing the CFSG-list help us, in practice?
To see some (easy) ways, first some general group theory...

E.g.: Consider the situation within the proof of the CFSG:
In a minl counterexample (some "unknown" simple G),
by ind, we may apply the CFSG-list to any $H < G$.
What kind of description of H does the CFSG-list give?

One elementary route: We can find a composition series:

$$1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_{n-1} \triangleleft H_n = H \ ,$$

where each successive quotient H_{i+1}/H_i is simple.
(By Jordan-Hölder (eg [Asc00, p 24]), up to re-ordering,
these quotients are independent of choice of series.)
By ind, each quotient is "known": on CFSG-list, or a $\mathbb{Z}_p$.
This "$\mathcal{K}$-group" description is a start; sometimes it suffices.

For a finer description, we can also apply the CFSG-list
to a certain "crucial" subgroup of H:

§1: Components—and the generalized Fitting subgroup

One notion of "crucial" arises in solvable groups:

Classically, the Fitting subgroup of any group X is:
$F(X) :=$ product of all nilpotent subnormal subgroups of X.

The solvable case is "self-centralizing" (e.g. [Asc00, 31.10]):
For solvable X, $C_X\big(F(X)\big) \leq F(X)$.
(And then it follows that $C_X\big(F(X)\big) = Z\big(F(X)\big)$.)
Here I call $F(X)$ "crucial", because of various consequences:

- E.g.: We could only get $F(X)$ trivial, when X itself is.
- Indeed $X/Z\big(F(X)\big)$ acts *faithfully* on $F(X)$.
- Further $X/F(X)$ induces *outer* automorphisms of $F(X)$.
  (So if $\mathrm{Out}\big(F(X)\big)$ is small, $F(X)$ is "most of" X.)

Bender gave an analogous essential subgroup—for general X...

The composition factors of nilpotent $F(X)$ are of prime order.
What if we instead use (nonabelian) simple factors?
The appropriate analogue of $F(X)$ turns out to be:

E(X) := prod of all quasisimple subnormal subgroups of X.

Here L quasisimple means $L = [L, L]$ with $L/Z(L)$ simple.

A quasisimple subnormal L is called a *component* of X.

($E(X)$ was developed from the *layer* of Gorenstein-Walter.)

See e.g. [Asc00, 31.7,31.12] for some properties, such as:

- components commute with each other (and with $F(X)$);
- their product in $E(X)$ is central (direct mod $Z\big(E(X)\big)$);
- further X acting on $E(X)$ permutes the components.

The *generalized* Fitting subgroup is $F^*(X) := E(X)F(X)$.
Now we get "self-centralizing"—for general X:
Property: (eg [Asc00, 31.13]) $C_X\big(F^*(X)\big) \leq F^*(X)$.

And as before, we get "crucial"-consequences for $F^*(X)$;
e.g., $X/F^*(X)$ induces outer automorphisms of $F^*(X)$.

With these structures in hand,
we return to our earlier inductive CFSG setup of $H < G$.
We can now describe $E(H)$: for each component L,
the simple quotient $S := L/Z(L)$ is on the CFSG-list.
(And L is limited by the (known!) *Schur multiplier* of S.)
Furthermore the action of $H/F^*(H)$ on $E(H)$ is restricted:

- via permutations of (sets of isomorphic) components L;
- with kernel inducing elements of $\mathrm{Out}(L)$ (hence $\mathrm{Out}(S)$).

This more explicit description of H
is often more useful than just a composition series for H.

A group X with $F^*(X) =: S$ simple is called *almost*-simple.
Then X is an extension of S, by some subgroup of $\mathrm{Out}(S)$.

The above focus on $\mathrm{Out}(S)$, for simple S on the CFSG-list,
also motivates the outer-aut analysis in the next section.

§2: Outer automorphisms of simple groups (Schreier)

The classical conjecture of Schreier is:
For (nonabelian) simple G, $\mathrm{Out}(G)$ is solvable.
This was checked for known G, as each was discovered;
so when CFSG was proved, it instantly became a theorem.
(It was also used—inductively—in parts of the CFSG proof.)

As an easy case of explicitly applying the CFSG-list,
let's trace the steps in that verification (eg [GLS98, 7.1.1]):

Alternating groups: See e.g. [GLS98, 5.2.1] for:
$\mathrm{Out}(A_n) \cong \mathbb{Z}_2$—except $\mathbb{Z}_2 \times \mathbb{Z}_2$ for $n = 6$.
These are certainly solvable (even elem abelian 2-groups).

Sporadic groups: See e.g. [GLS98, Table 5.3.a–z] for:
For G sporadic, $\mathrm{Out}(G)$ has order: 1; or 2 (in 12 cases).
Again solvable (and indeed again elem abelian 2-groups).

But don't conjecture the latter in general; because...

Lie type groups: (Only case with interesting structure...)
For G of Lie type, see e.g. [GLS98, 2.5.12,1.15.7]:
(The "diagonal-field-graph" theorem:)
$\mathrm{Out}(G)$ has normal subgp D, with quotient $F \times \Gamma$; where:

- D induces *diagonal* automorphisms,
- F induces *field* automorphisms, and
- Γ induces *graph* automorphisms.

Each of the 3 groups is abelian; so $\mathrm{Out}(G)$ is solvable.

(Some further details—summarized very roughly:)
In fact D is a subgroup of the (abelian) Cartan subgroup H;
bounded by the *fundamental group* [Car89, Sec 7.1]
($\mathbb{Z}_n$ for linear L_n; otherwise order ≤ 4).
Further F is cyclic: the Galois group of $\mathbb{F}_{p^a}$ has order a.
And for Γ, we saw Dynkin-diagram symmetries of order 2 or 3.
(In twisted cases, details can be more complicated...)

Exer 1.5: Explore various automorphism types;
e.g. in some linear/classical cases.

Some samples—using standard isomorphisms among some small cases; cf. [ALSS11, p 261], [Wil09, 3.11,3.12].

1) Transpositions in S_5 exhibit $\mathrm{Out}(A_5)$.
But A_5 can also be regarded as $L_2(4)$ or $L_2(5)$:

- The automorphism for $L_2(4)$ has field type, in this view;
- whereas viewed for $L_2(5)$, it has diagonal type.

(Explain...)

2) Transpositions in S_6 give one element of $\mathrm{Out}(A_6)$;
for another: view A_6 as M'_{10} (pt stab in M_{11}—[C$^+$85, p 4]).

But A_6 can also be regarded as $L_2(9)$ or $Sp_4(2)'$.

- For $L_2(9)$, automorphisms have diagonal and field types.
- For $Sp_4(2)'$, they come from: $Sp_4(2)$; and graph type.

(Explain...)

Remark: Many uses of CFSG really only need Schreier(!)

Afterword: Determining multiply-transitive groups

Just briefly: another “instant” consequence of CFSG.

For *permutation groups*, see e.g. Cameron [Cam99].
An early view of consequences of CFSG is in [Cam81];
we'll just extract a rapid overview:

Classical problem: Determine multiply-transitive groups.

(Since M_{12} and M_{24} are 5-transitive, also have:
Sub-problem: Show 6-transitive forces A_n or S_n.
In fact, work of Wielandt, Nagao, O'Nan
had reduced the sub-problem to the Schreier Conjecture.)

For main problem: 2-trans suffices (as includes higher trans).
Burnside gave the basic reduction for a 2-trans group H:
H has unique minl normal sgp N—el ab, or nonab-simple.
The latter could then be checked, using the CFSG-list.

Abelian-N: by Huppert for H solvable; while Hering and others used CFSG-list for simple in H_α—see e.g. [Lie87a, App].

Simple-N was handled by work of
Maillet, Howlett, Curtis-Kantor-Seitz.

Here is a rough summary of the result [Cam81, 5.3]:

The simple-N subcase of 2-trans H must be one of:

(alternating:)

- A_n on points of natural perm repn
- A_7 on 15 (from "sporadic" containment in $L_4(2)$)

(Lie type:)

- $L_n(q)$ on points of proj space of natural module
- Lie rank 1, on points of projective line
- $Sp_{2d}(2)$ on orthogonal forms (types $+$ and $-$)
- $L_2(11)$ on 11 (as opposed to usual 12 of proj line)

(sporadic:)

- Mathieu groups; HS on 176; Co_3 on 276

For the sub-problem, it follows that:
Only A_n and Mathieu groups are 4- or 5-transitive.

Another (quick) drive-by shouting:
A further instant consequence of Schreier Conj (hence CFSG)
was the Hall-Higman reduction [HH56]
for the *restricted Burnside problem*.
The reduction was in turn used by Zelmanov [Zel91]
in his celebrated solution of that problem.

THANKS!

Lecture 2: An outline of the proof of the Classification of Finite Simple Groups (version of 25sept2015)

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Overview of the talk:

Introduction: *examples*, motivating odd versus even cases.
0: Starting: The Odd/Even Dichotomy Theorem.
(+ small/generic subcases: the "grid")
1: Odd Case: Handled via "standard" components.
2: Even Case: Handled via a *tri*chotomy of subcases.

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`
(cf. also lectures at `.../talkn.pdf` and `.../talkst.pdf`)

Foreword

This lecture samples Chapters 0–2 of the outline [ALSS11].
Why did those authors give such an extensive exposition?

- while memory is still available...
- Davies [Dav05] on long proofs
  quoted Aschbacher roughly "no published outline"...

Actually, Gorenstein had started an outline
(intro [Gor82]; odd case [Gor83]); but couldn't finish—
—as "quasithin" (in even case) done later [AS04c].
So, A-L-S-S completed his outline with [ALSS11].

Today's goal: overview main *case divisions*, as in [ALSS11].
(For "original" CFSG. Later approaches? See end of talk.)

Introduction: examples of odd and even cases

First, to suggest "characteristic" for a simple group G:
recall (lec1) the eventual list of examples (i.e. conclusions):

CFSG Theorem: (E.g., [ALSS11, Thm 0.1.1])
A (nonabelian) finite simple group G is one of:
i) An alternating group $A_n (n \geq 5)$.
ii) A group of Lie type.
iii) One of 26 "sporadic" groups.

Further we'd said "most" simple G are in (ii)— of Lie type;
i.e. matrix groups over a finite field, of order p^a (p prime).

So, a "typical" simple G has a characteristic p. Can ask:

- Why distinguish odd p, versus even (i.e. $p = 2$)?
- Where in G can we see this odd/even distinction?

The "pre-history" of the CFSG begins to suggest answers.
E.g. looking at *involution centralizers*. They arise, since:

Feit-Thompson Theorem [FT63]:[3]
A group of odd order is solvable.

Thus, a simple G has elements t of order 2 ("involutions").
Further: centralizer $C_G(t) <$ simple G.
So, might we proceed by applying induction to $C_G(t)$?

Brauer-Fowler Theorem [BF55]:
Fix some group H, with an invol t in the center $Z(H)$.
Then only *finitely* many simple G can have $C_G(t) \cong H$.

This finiteness makes feasible an approach to CFSG:

- determine possible H (using induction for structure);
- then for each H, determine simple G with $C_G(t) \cong H$.

And such analysis is, roughly, how the CFSG was proved.

---

[3] Ideas here also led to *signalizer functors*—cf. later in the talk.

But just for today's expository purposes:

The basic odd/even case division will emerge,
when we next glimpse $C_G(t)$ in typical examples G.
Indeed it suffices to examine in "nearly" simple GL_n—
—where we can just draw pictures of square matrices:

Example—Odd Case: Take p odd, in $G := GL_n(p^a)$.
For involution, take t diagonal, with some -1's:

$$t := \left(\begin{array}{c|c} -I_k & 0 \\ \hline 0 & I_{n-k} \end{array} \right)$$

Any commuting g preserves (± 1)-eigenspaces; so get picture:

$$C_G(t) = \left(\begin{array}{c|c} GL_k(q) & 0 \\ \hline 0 & GL_{n-k}(q) \end{array} \right)$$

Next—let's *abstract* one particular feature from this example:
$C_t := C_G(t)$ has (nearly-)simple normal subgroup(s).
(We get this feature also in the other odd-char Lie-type G;
and in alternating groups, as well as some sporadics.)
Indeed recall the language of lec1a:
these are components (quasisimple, subnormal) of C_t.
Exer 2.1: Find such components of suitable C_t—
—in some small (but large-enough) G of the given types.

This suggests how to abstractly define our "odd case":
We say G is of *component type*,
if for some inv t, $C_t/O_{2'}\big(C_t\big)$ has a component.
Or, can use terminology: C_t has a 2-*component*.
(Why work mod $O_{2'}\big(C_t\big)$? See pf of Dichotomy Thm, later.)

From now on, component type gives (most of) our Odd Case.

The examples look very different, when we instead take $p = 2$:
Example—Even Case: Take $p = 2$; i.e. now $G := GL_n(2^a)$.
Here $1 = -1$: so our earlier diagonal t has order 1, not 2!
Instead take t "smallest": single 2×2 Jordan block; conj to:

$$t := \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & I_{n-2} & 0 \\ \hline 1 & 0 & 1 \end{array}\right)$$

Check by straightforward computation that:

$$C_t := C_G(t) = \left(\begin{array}{c|c|c} * & 0 & 0 \\ \hline * & GL_{n-2}(2^a) & 0 \\ \hline * & * & * \end{array}\right)$$

Here C_t is a *semi*-direct product[4] UL, where:

$$U := \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline * & I_{n-2} & 0 \\ \hline * & * & 1 \end{array}\right), \; L := \left(\begin{array}{c|c|c} * & 0 & 0 \\ \hline 0 & GL_{n-2}(2^a) & 0 \\ \hline 0 & 0 & * \end{array}\right)$$

The nearly simple $GL_{n-2}(2^a)$ in L here might seem similar to components in the earlier odd-case example; but instead:

(Check) L is *not* normal in C_t !

Indeed "only" the 2-subgroup U is normal; in fact:

C_t has no subnormal quasisimple (or odd-order) subgroup.

(Get same in other even Lie type; and in a few sporadics.)

Exer 2.2: Check this feature, in some small even-Lie cases.

[4] In Lie theory (lec1): the *Levi decomposition* of the parabolic C_t.

In the language of lec1a, the above feature is:

$$F^*(C_t) = O_2(C_t).$$

Where we recall:

$F^*(X) := E(X)F(X)$; with

$E :=$ prod of subnormal quasisimple sgps (components),

$F :=$ prod of subnormal prime-power (nilpotent) sgps.

This suggests how to abstractly define our "even case":
We say G is of *characteristic* 2 *type*,
if $F^*(N) = O_2(N)$ for all 2-local subgroups N.
(In fact: suffices, if true for all C_t—[ALSS11, B.1.6])
From now on, characteristic 2 type gives our Even Case.

Now—leaving aside our odd/even examples above:
How might we abstractly show that simple G should be
either (component-) or (characteristic 2-) type?

§0: To start—prove Odd/Even Dichotomy Thm

"Small" case $m_2(G) \leq 2$ done early (e.g. [ALSS11, 1.4.6]).
So the "generic" case is $m_2(G) \geq 3$; where we get:

Dichotomy Theorem: (Cf. 0.3.10 or B.3.5 in [ALSS11])
Assume G is simple, with $m_2(G) \geq 3$. Then:
either G is of component type,
or G is of characteristic 2 type.

The proof is elementary, and fairly short (4 pages)—
—modulo *assuming* two "basic" early-70s theorems.
And ideas are re-used repeatedly, in the rest of CFSG. So:

Sketch: If $\boxed{\text{component-type case}}$—done; so assume that *fails*.
Then for each inv t, setting $\overline{C}_t := C_t/O_{2'}(C_t)$, we have:

$$F^*(\overline{C}_t) = O_2(\overline{C}_t).$$

NEED: char 2 type; so want above, "unbarred"—for C_t itself.
Thus: suffices, to get $O_{2'}(C_t) = 1$; for then $\overline{C}_t = C_t$.

STRATEGY: Extend the function $\theta : t \mapsto O_{2'}(C_t)$
to larger elem ab $\langle t, u\rangle$ etc ... then force larger-$\theta(-) \equiv 1$.

(STEP 1: Balance—leading to signalizer functor:)

Using "no 2-components", props of $F^* \Rightarrow$ *balance*:

$$\theta(t) \cap C_u = \theta(u) \cap C_t, \text{ for } t \in C_u.$$

Then θ is called a *signalizer functor*. (Why? because:)

(STEP 2: Completeness—leading to graph components:)

As $m_2(G) \geq 3$, we can quote the (standard)

Signalizer Functor Theorem [ALSS11, 0.3.14], to get:

- For elem ab C, $\theta(C) := \langle \theta(t) : t \in C^{\#}\rangle$ is still a $2'$-group!
  ("signalizer"-valued; $+$funct*or*: natural w.r.t. $\leq$ and conj)
- *complete*: $\theta(t) = \theta(A) \cap C_t$ for $t \in A$ of rank ≥ 3.
- Further $\theta(A) = \theta(B)$ for all B of rank ≥ 2 in A.

From last—see θ is constant on connected-comps of *graph* Γ:
with vertices like A; edges, by intersecting in a B.

Now get two cases for possible connectivity:

$\boxed{\text{The disconnected case:}}$

Here, quote the **Strongly Embedded**[5] **Theorem** (Bender-Suzuki [ALSS11, 0.2.3]) to get:

G is of Lie type, and Lie rank 1, in characteristic 2.[6]

Such G does have characteristic 2 type. (So here $\theta \equiv 1$.)

$\boxed{\text{The connected case:}}$

Here, $\theta(A)$ is constant e.g. on the conjugacy class of A. So G normalizes $\theta(A)$; hence by simplicity, $\theta(A) = 1$. Thus $\theta(t) = 1$ for $t \in A$;[7]and we get $\theta \equiv 1$ here also. (That is, G has component type; as required.)

This completes the proof of the Dichotomy Theorem. $\square$

---

[5]Means that $N_G(\mathrm{Syl}_2)$ intersects conjugates in odd order. Get *unique* maxl 2-local over Sylow; so expect (lec1) *one* node in Dynkin diagram...

[6]The "Bender groups": $L_2(q)$, $U_3(q) = {}^2A_2(q)$, $Sz(q) = {}^2B_2(q)$.

[7]To see every t is in some A, cf. proof of 0.3.22 in [ALSS11].

Note: The proof really has a *tri*chotomy structure:
$\{$ component type, disconnected, connected $\}$.
(It just happened for $p = 2$ that the latter two co-incided...)

The notions of balance and signalizer functors
(with completeness/connectedness leading to trichotomy)
were adjusted/re-used in many ways, throughout the CFSG.

The Dichotomy Thm gives a basic Odd/Even case division.
But recall its proof uses $m_2(G) \geq 3$...

Historically, $m_2(G) \leq 2$ in effect covered
the Small Subcase of the Odd Case.
(So the union of $m_2(G) \leq 2$ and component type
is called "Gorenstein-Walter type" (GW) in [ALSS11].)

To re-use the same method in the Even Case,
we'll need a further small/generic case subdivision:
Recall that characteristic 2 type focuses on 2-local subgroups;
so we consider *odd* p-ranks—in 2-locals:

$m_{2,p}(G) := \max\{m_p(H) : H \text{ is a 2-local subgroup of } G\}$.

$e(G) := \max_{p \text{ odd}} m_{2,p}(G)$.

The Small Even Subcase is given by $e(G) \leq 2$ ("quasithin").
(As was already clear in Thompson's N-group work [Tho68].)

Thus the original CFSG treats a "grid" of four subcases:

	(Odd Case) *GW* type	(Even Case) characteristic 2 type
small	$m_2(G) \leq 2$	$e(G) \leq 2$ (quasithin)
generic	component type	$e(G) \geq 3$

So we now turn to how the various subcases were handled.

§1: The Odd Case—via “standard components”
(Still skipping “small” $m_2(G) \leq 2$—e.g. [ALSS11, 1.4.6]):
So “generic” G now has component type—and $m_2(G) \geq 3$.
Thus for some involution t, C_t has a 2-component L.

Fundamental contributions of Aschbacher led to
the following (idealized!) strategy (cf. [ALSS11, 1.1.1]):

- show some L is in fact a component (i.e. quasisimple);
- show some suitably-maximal L has “standard form”;
- show G is (usually) a larger group of same type as L.

In a little more detail:
Defn: L is *standard* in G if: (settting $H := C_G(L)$)

- L commutes with none of its G-conjugates;
- $H \cap H^g$ has odd order for $g \in G \setminus N_G(L)$.[8]

(Motivation? recall Odd Case Example—now with $k = 2$)
Exer 2.3: Find std, in further odd Lie/alt/spor examples.

(I now skip over the Unbalanced Group Theorem,
and the proof of Thompson's B-Conjecture—roughly,
these lead from a 2-component to a component.
Cf. [ALSS11, Secs 1.6–1.8].)

[8] H is *tightly* embedded in G; a relaxation of strongly embedded.

The **Standard Component Theorem**
of Aschbacher [ALSS11, 1.8.12] shows for our G
(assuming proof of the B-Conjecture above) that:

- either G is one of several smallish Odd cases (so done!);
- or for some t, C_t has a component L which is standard.

Thus our Odd Case is reduced to:

- considering each possible quasisimple L;
- and determining which G can have that L standard.

The latter is called the standard-form problem for L.
Expect: G to be a larger group of same "kind" as L.[9]

The various standard form problems were handled by $\sim$1979;
in papers by more than 20 different authors.
(See [ALSS11, Secs 1.9–1.10] for a list.)
That completed the treatment of the Odd Case.

So we turn to the Even Case...
...where we will roughly adapt the same ideas, for re-use!

[9] Again recall the earlier Odd Case example.

§2: Even Case—via a trichotomy of subcases

In this section, G has characteristic 2 type (and $m_2(G) \geq 3$).
So for each involution t, $F^*(C_t) = O_2(C_t)$.
(Indeed the same property holds for all 2-local subgroups.)

In desired conclusion groups G of even Lie type,
$e(G)$ is roughly Lie rank—cf. 2-local C_t in Even Case Example.
Exer 2.4: Check they agree, in some other even-Lie cases.

The "small" case is now $e(G) \leq 2$ (quasithin).
Quasithin simple G were (eventually) classified by Aschbacher-Smith in [AS04c].

So henceforth: assume "generic" case $e(G) \geq 3$.
Can we employ some version of signalizer functors?

In Odd Case, t had order 2—different from char of odd Lie G.
So in Even Case, for even-Lie G, take u of *odd* prime order p;
do we get "p-components" etc?

Motivate? Revisit Odd Case Example $(-1) \mapsto$ (p-th root of 1).
Exer 2.5: Find p-comps in C_u, in some other even-Lie cases.

So with the notion of p-components in C_u:
- again get signalizer functors; balance; completeness;
- with graph components, again leading to a trichotomy:

$$\{\ p\text{-component type}\ ,\ \text{disconnected}\ ,\ \text{connected}\ \}$$

(The latter two now distinct—as no *odd* Bender-Suzuki Thm.)

In summary, these "same" elementary (!) arguments give:

WEAK Trichotomy Theorem [ALSS11, 2.2.1]:
For G simple of char 2 type, with $m_2(G) \geq 3$, $e(G) \geq 3$, there is suitable odd p giving one of:

- p-uniqueness type (disconnected);
- p-component type;
- characteristic $\{2, p\}$ type (connected).[10]

Good...but not enough. More sophisticated analysis leads to:

STRONG Trichotomy Theorem: [ALSS11, 2.3.9]
For G simple of char 2 type, $m_2(G) \geq 3$, $e(G) \geq 3$,[11] there is odd p with one of: (rough defns on next page...)

- p-Preuniqueness Case;
- p-Standard Type;
- $GF(2)$ Type.[12]

---

[10] Here Klinger-Mason methods [ALSS11, B.9.2;p27] give $GF(2)$-type below. Ex: *Har*; e.g. Franchi-Mainardis-Solomon [FMS08], with $p = 5$.

[11] Done for $e(G) \geq 4$ by Gorenstein-Lyons [GL83]; and Aschbacher's treatment of $e(G) = 3$ [Asc81a, Asc83a] includes this trichotomy.

[12] In $e(G) = 3$, Aschbacher allowed more general $GF(2^n)$ type.

How were these three branches handled? In rough summary:

(Branch designed by G-L to include G over $\mathbb{F}_2$ plus sporadics:)
$GF(2)$ **type:** roughly, as in Lie-type groups over $\mathbb{F}_2$;
have inv t with $O_2(C_t)$ of *symplectic type*.[13]
Exer 2.6: Check this, in some other Lie/$\mathbb{F}_2$ cases.
This situation had *already* been handled $\sim$1978, by:
Aschbacher; Timmesfeld; Smith; Stroth; ...
(See [ALSS11, 7.0.1]; for $GF(2^n)$ type, see [ALSS11, 7.5.2].)

(Branch designed for most conclusions: G of even Lie type.)
Standard type: roughly, p-component is in fact component;
with strengthened standard-form properties.
Unlike original standard form with many authors/papers,
here standard type done in one work Gilman-Griess [GG83].

[13] "Close to" extraspecial: $X' = \Phi(X) = Z(X)$ order 2.

(Branch designed for final contradiction of CFSG.)
Preuniqueness Case: roughly, an analogue of:
stabilizer of a connected component lies inside a 2-local.

First, get to "full" Uniqueness Case
(a 2-local which is "almost" strongly p-embedded).
This was done by Aschbacher-Gorenstein-Lyons [AGL81].

Then, the Uniqueness Case was eliminated
by Aschbacher in [Asc83b, Asc83c].

That completed the proof for the Even Case—and the CFSG.

Afterword: Later approaches to CFSG?

The bulk of the lecture has been about:
1) the "original" CFSG...

We'll now comment (briefly) on some later alternative routes:

2) "Revisionism"—Gorenstein-Lyons-Solomon [GLS94] etc:

- goal of "all in one place"
- new: *some* 2-components allowed, in "even *type*" (so vertical partition in grid shifted a bit left—cf orig-grid);
- 6 of expected 10 volumes out so far;
- for quasithin, now using [AS04c];
- for Uniqueness Case? (Stroth 2009 preprint@website).

3) Meierfrankenfeld Stellmacher Stroth—char-p methods (overview: [MSS03]; for status, see Ulrich's website)

- avoid non-char prime, incl: odd-std form; quasithin; grid!?
- get *structure* of parabolics; identify G so generated.

4) Aschbacher (e.g. [Asc15]), via *fusion systems*:

- use category given by (2-Sylow, conjugacies)...
- classify simple fusion systems; then the corresp groups
  (For 2-comp type, this avoids problems with $O_{2'}$ ...)

5?) You young folks should find *further* approaches!

Lecture 2a: Applying CFSG toward Quillen's Conjecture
(version of 25sept2015)

Stephen D. Smith
U. Illinois–Chicago

Venice Summer School
7-18 September 2015

Overview of the talk:
Introduction: The complex $\mathcal{S}_p(G)$ and Quillen's Conjecture
1: Quillen-dimension and the solvable case
2: The reduction for the p-solvable case
3: Other uses of CFSG in Aschbacher-Smith

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

Intro: The complex $\mathcal{S}_p(G)$ and Quillen's Conjecture

Starting $\sim$ late 1970s, "subgroup complexes" emerged as an area of common interest to researchers in:

- finite group theory,
- combinatorics,
- algebraic topology.

For let $\mathcal{C}$ be some set of subgroups of finite G;
and let $|\mathcal{C}|$ denote the set of inclusion-*chains* in $\mathcal{C}$.
Since a subset of a chain is still a chain,
we see that $|\mathcal{C}|$ is in fact a simplicial complex. (...topology!)
Indeed if $\mathcal{C}$ is closed under conjugacy, $|\mathcal{C}|$ admits a G-action.

For brevity, we will now write just $\mathcal{C}$ also for this complex $|\mathcal{C}|$.
(Excluding $1, G$ from $\mathcal{C}$ avoids "obvious" contractibility.)

In [Smi11] I surveyed much of the complexes-literature...

The p-structure of G (e.g. Sylows) is seen in the complex (of):

$$\mathcal{S}_p(G) := \{ \text{ non-trivial } p\text{-subgroups of } G\}.$$

Indeed Brown ∼1975 (e.g. [Smi11, 0.0.1]) established a "homological Sylow theorem" (for Euler characteristic χ):

$$\chi\big(\mathcal{S}_p(G)\big) \equiv 1 \pmod{|G|_p}$$

Quillen's influential paper [Qui78] studied more properties.

E.g. he extended (e.g. [Smi11, 6.2.1]) arguments of Brown, generalizing the Steinberg module of a Lie type group:

$$\tilde{L}\big(\mathcal{S}_p(G)\big) \text{ is a (virtual) } \textit{projective} \text{ module}$$

for $\tilde{L}$ the alternating sum of (reduced) homology groups.

(Webb et al developed the area further—e.g. [Web87].)

Indeed among further topological properties of the complex:

Quillen also showed, fairly easily, that: (e.g. [Smi11, 3.3.5])
If $O_p(G) > 1$, then $\mathcal{S}_p(G)$ is contractible.
And (boldly!) conjectured the converse: (e.g. [Smi11, 3.3.8])
Quillen Conj: If $O_p(G) = 1$, $\quad \mathcal{S}_p(G)$ is not contractible.

Quillen himself established some important special cases:
Ex: solvable; Lie type in char p; ... cf. [Smi11, 8.4.2].
Later authors established further special cases;
Ex: (known) simple—Aschbacher-Kleidman [Smi11, 8.4.3].
The most general result so far is:

Thm: (Aschbacher-Smith [AS93]) For $p > 5$:
Conj holds—unless $E(G)$ has certain unitary components.

A theme: exploit the (central-)product structure of $E(G)$—
—which gives a *join* of the complexes;
then apply CFSG to identify and analyze the components.

Work leading to this result is discussed in [Smi11, Ch 8];
and below we will sketch some of that exposition.

§1: Quillen-dimension and the solvable case

For non-contractibility, it suffices to get nonzero homology. But where? e.g. in what dimension should we find it?

We note that Quillen in fact works with the smaller poset:

$\mathcal{A}_p(G) := \{$ non-trivial *elem abel* p-subgroups of G $\}$;

as he can—since he gets (e.g. [Smi11, 4.2.4]) that:

$\mathcal{A}_p(G)$ is homotopy equivalent with $\mathcal{S}_p(G)$.

So: no homology for $\mathcal{S}_p(G)$ in dim above that of $\mathcal{A}_p(G)$.

But often, can find homology in exactly that dimension:

Indeed we say G has *Quillen-dimension* (at p),

if $\tilde{H}_{d-1}\big(\mathcal{A}_p(G)\big) \neq 0$, where $d := m_p(G)$.

This notion is crucial for the Aschbacher-Smith approach.

Next: How can we find homology ?
  i.e. what structures exhibit $\tilde{H}_{d-1} \neq 0$?
Note as top dimension, it suffices to just find *cycles*.

Easiest: sphere S^{d-1}—as join of d copies of $S^0 = \{\bullet\, \bullet\}$.
Ex: $p = 2$; in Thompson Dihedral Thm: [ALSS11, B.1.7]
  For A elem ab 2^d, faithful on a (solvable!) $2'$-group L,
  we can find direct product of d dihedral $A_i L_i \cong D_{2q_i}$,
  with A the product of the A_i of order 2.
Each $\mathcal{A}_2(A_i L_i)$ has $q_i \geq 3$ disconn vertices (so some S^0);
  thus the join-complex for the product has copies of S^{d-1}.
Exer 2.7: Spheres for prod of nonab groups of odd order pq?

This theme of faithful action of A on L
  will be prominent in what follows...

Quillen considers such an LA-situation: (cf. [Qui78, 11.2])
For A elem abel p^s, faithful on solvable p'-group L,
he shows $\tilde{H}_{s-1}\big(\mathcal{A}_2(LA)\big) \neq 0$.
(Thus in our terminology: LA has Quillen-dimension.)
(Alperin's later method [Smi11, 8.2.9] essentially reduces
to (p on q-gp) analogues of structures in Thompson's thm.)

Quillen then deduces the Conjecture, for G solvable;
with $O_p(G) = 1$, and A elem abel of rank $d := m_p(G)$.
From $O_p(G) = 1$, we get that $F(G)$ is a (solvable) p'-group.
So A is faithful on $L := F(G)$.
Taking d as "s" above, we get $\tilde{H}_{d-1}\big(\mathcal{A}_p(LA)\big) \neq 0$.
By *top*-dimension, these cycles also give homology for G:
that is, $\tilde{H}_{d-1}\big(\mathcal{A}_p(G)\big) \neq 0$.
Thus Quillen obtained (cf. [Smi11, 8.2.5]) that:
The Conj holds for solvable G (indeed G has Quillen-dim).

§2: The reduction for the p-solvable case

By the early 1990s, experts were aware that the solvable result could be extended to the case of *p-solvable G*.

This means composition factors can be p-groups, or p'-groups. (Note for $p > 2$, the latter need not be solvable: Ex: Suzuki groups $Sz(2^{\text{odd}})$ are simple $3'$-groups.)

From $O_p(G) = 1$, now get $F^*(G) = E(G)F(G)$ a p'-group; but nontrivial $E(G)$ (hence $F^*(G)$) would be non-solvable.

Again A of rank $d = m_p(G)$ is faithful on $L := F^*(G)$; so if suffices to get A faithful on some solvable $L_0 \leq L$.

I think no one had explicitly claimed the p-solvable result; it is quoted as well-known in [AS93, 0.5], with a proof indicated via 1.6 (using 0.10) there.

The reduction to L_0 was, inadvertently, omitted in 0.10; a standard argt was later supplied as [Smi11, 8.2.12].

We sketch that reduction—as (one) appl of CFSG in [AS93].

Let $L_i (i \in I)$ denote the components; group them in A-orbits.
Define L_0 as the (nilp) prod of $F(G)$—with Sylows S_i of L_i;
chosen so that $N_A(L_i)/C_A(L_i)$ acts faithfully on S_i. I.e. we get:

$$N_A(L_i) \le N_A(S_i); \text{ with } C_{N_A(L_i)}(S_i) \le C_A(L_i).$$

Then L_0, chosen in this way, will work:

If $a \in C_A(L_0)$, it is in each $C_A(S_i)$, hence in $N_A(L_i)$;
i.e. in $C_{N_A(L_i)}(S_i) \le C_A(L_i)$. Thus, centralizes $L = F^*(G)$.
As A faithful on L, $a = 1$. Hence A is faithful on L_0.

Thus it suffices to choose, in each L_i, a suitable S_i.

If $N_A(L_i) = C_A(L_i)$, *any* Sylow S_i has both conds above.

So we may assume $N_A(L_i) > C_A(L_i)$.
Setting $N_A(L_i)^* := N_A(L_i)/C_A(L_i)$,
we have nontrivial $N_A(L_i)^*$ in $\mathrm{Aut}(L_i)$—even in $\mathrm{Out}(L_i)$.
In a moment, we'll apply CFSG—and knowledge of $\mathrm{Out}(L_i)$.

But first, we locate a suitable candidate for S_i:

Any q-Sylow Q has $|L_i : N_{L_i}(Q)|$ conjugates—coprime to p.
So for each q, the p-group $N_A(L_i)$ normalizes some Sylow.
Any Sylow S_i so chosen has first cond: $N_A(L_i) \leq N_A(S_i)$.

So now suffices to get second cond. (Will take longer...)
We begin by choosing S_i with some additional restrictions:
As $N_A(L_i) > C_A(L_i)$, there is some q
with S_i not centralized by $N_A(L_i)$.
That is, $N_A(L_i)^*$ does not centralize this S_i.

We next apply the CFSG, to determine possible L_i;
and for known L_i, examine $N_A(L_i)^* \leq \mathrm{Out}(L_i)$:

Recall $N_A(L_i)^*$ is a nontrivial p-group; whereas L_i is a p'-group.

This eliminates L_i alternating or sporadic:
for there we saw outer automorphisms have order 2 or 3,
and those primes do divide the order of L_i.

So L_i has Lie type.
The same argument eliminates graph automorphisms—
—order 2 or 3 (the $3'$-group $Sz(2^{\text{odd}})$ has none);
it also eliminates diagonal auts—primes as in Cartan H.

Thus only field automorphisms arise;
we saw these are cyclic (Galois group of finite field),
so we get elem ab $N_A(L_i)^*$ of order exactly p.

Recall we chose with $N_A(L_i)^*$ not centralizing S_i.
As $N_A(L_i)^*$ has prime order, we conclude $C_{N_A(L_i)^*}(S_i) = 1$;
that is, we get second cond: $C_{N_A(L_i)}(S_i) \leq C_A(L_i)$.
(**Exer 2.8:** Give some small examples of such A, L_i, S_i.)

This completes the reduction from L to solvable L_0. $\square$

§3: Other uses of CFSG in Aschbacher-Smith

We close with a few main features [AS93];
also see the discussion in [Smi11, Ch 8].

Consider Quillen Conj for general G; with hyp $O_p(G) = 1$.

The p-solvable case above leads to $O_{p'}(G) = 1$.
This gives $F(G) = 1 = Z\big(E(G)\big)$.
So $F^*(G) = E(G)$ is direct product of *simple* components.

This suggests the main overall approach:
In $E(G)$ (known by CFSG), find nonzero product homology;
and show this "survives" in homology for G.
It turns out that Quillen-dimension is very useful for this...

Thus Section 3 of [AS93] shows roughly that:
"most" (almost-)simple groups have Quillen-dimension.
The major exceptions are:

- Lie type groups in the same characteristic p;
- certain unitary groups $U_n(q)$ with $q \equiv -1 \pmod{p}$.

Of course the CFSG-list is used for the groups to analyze.
Method: find p, q-spheres S^d as above, via known subgp-str...

To elim $E(G)$ with comp L, where LA has Quillen-dimension;
i.e., faithful A, of maximal p-rank $m_p(LA)$:
One can (carefully!) apply induction to $C_G(LA)$:
arrange $O_p\big(C_G(LA)\big) = 1$; then, by ind, nonzero homology.
The product of that, with our (top) homology of LA,
gives homology of product group $C_G(LA)LA$. For G?
If in image of boundary map, would come from el ab $C \times B$:
with $C \leq C_G(LA)$, and $A < B \leq LA$; but A was maxl! $\square$

So from now on, each comp L of $E(G)$ is a "non-QD" group.

For these cases, we use a method due to Robinson [Rob88].
Given X q-hyperelementary (cyclic extended by q-group), easy:
for fixed-pts get $\tilde{\chi}\big(\mathcal{S}_p(G)\big) \equiv \tilde{\chi}\big(\mathcal{S}_p(G)^X\big) \pmod{q}$.

IF we had contractibility: then LHS=0; so RHS divisible by q.
For $q = 2$, can construct an X—where RHS is instead *odd*.

Section 5 of [AS93] roughly studies CFSG-list to show:
For L a non-QD group (*except* those unitary groups!),
can find 2-hyperelementary Y with $\tilde{\chi}\big(\mathcal{S}_p(L)^Y\big) = \pm 1$.
(Mostly, fix-pts=$\emptyset$ ($\tilde{\chi} = -1$); or, just 2 (S^0, $\tilde{\chi} = 1$).)

Finish? Assume $E(G)$ has no non-QD unitary component.
Using prod of Y above, build a suitable 2-hyperelementary X;
can show $\mathcal{S}_p(G)^X = \mathcal{S}_p\big(E(G)\big)^X$,
with $\tilde{\chi}$ given by the product of the $\tilde{\chi}\big(\mathcal{S}_p(L)^Y\big)$.
We conclude $\tilde{\chi}\big(\mathcal{S}_p(G)^X\big) = \pm 1$.
So by the congruence above, $\mathcal{S}_p(G)$ has nonzero homology. $\square$

THANKS!

Lecture 3: (Failure of) Thompson factorization—etc

(version of 25sept2015)

Stephen D. Smith
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Venice Summer School
7-18 September 2015

Overview of the talk:
Introduction: Some forms of the Frattini factorization.
1: The Thompson Factorization—with $J(T)$ as "W".
2: Determining when that factorization can fail—(FF).
3: Pushing-up methods: FF-modules—in Aschbacher blocks.
4: Weak-closure factorizations: other "W".

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`
(Cf. earlier `www.math.uic.edu/~smiths/talkc.pdf`)

Foreword

This talk largely follows corresponding exposition (on CFSG) in [ALSS11], especially Sections B.6–B.8 there.

It also draws from expository material in Aschbacher-Smith [AS04a, AS04c] (on quasithin groups); especially from Chapters B, C, and E in [AS04a].

(An underlying theme: fundamental ideas of Thompson; with further developments by Aschbacher.)

Introduction—some forms of the Frattini Factorization

Take H a group (e.g. a p-local in simple G). Elementary:

If $H \geq X$ transitive on an H-orbit (say with repr α), then

$$H = X \cdot H_\alpha \ .$$

One case—**Frattini Argument** (E.g. [ALSS11, A.1.5(2)]):

If $N \trianglelefteq H$ with $P \in \mathrm{Syl}_p(N)$, then

$$H = N \cdot N_H(P) \ .$$

"Module" subcase: For H acting on "internal module" V:

If V elem.ab. p-group $\trianglelefteq H$, with $P \in \mathrm{Syl}_p\big(C_H(V)\big)$, then

$$H = C_H(V) \cdot N_H(P) \ .$$

(Refinements Z, W:) Continue with V, P as above. Notice:
For any $Z \leq V$, we have $C_H(V) \leq C_H(Z)$.
For any W *weakly closed* in P,[14] have $N_H(P) \leq N_H(W)$. So:

If V, P, Z, W are as above, then

$$H = C_H(Z) \cdot N_H(W) \qquad \text{(FA)}$$

This leads to later factorization forms, e.g. Thompson Fact; weak cl...

[14] means any H-conjugate of W, inside P, must in fact be W

The above form (FA) of Frattini (i.e. the setup "V, P, Z, W")
arises frequently in the analysis of p-local subgroups—
—especially in the following **standard situation:**

Consider G of char p type. Then each p-local H has:

$$O_p(H) = F^*(H) \text{—so that } C_H\big(O_p(H)\big) = Z\big(O_p(H)\big).$$

Now: further *assume* that H contains a Sylow T of G;
and take "Z" $= \Omega_1\big(Z(T)\big)$. (or just $\leq$)
Notice $Z \leq C_H(T) \leq C_H\big(O_p(H)\big) = Z\big(O_p(H)\big)$.
So as our module, take "V" $:= \langle Z^H \rangle$:
For it is $\trianglelefteq H$; and elem.ab., since $\leq Z\big(O_p(H)\big)$ by above.
(Even get V is "p-reduced": i.e. $O_p\big(H/C_H(V)\big) = 1$.)[15]

Next: how choose a "W" w.cl. in "P" $\in \mathrm{Syl}_p\big(C_H(V)\big)$?
Here Thompson's deeper insights will come into play...

[15] See e.g. [AS04a, B.2.13] for this fact.

§1: Thompson Factorization—with $J(T)$ as "W"

Thompson introduced (e.g. [ALSS11, B.6.4]):

$J(T) := \langle$ all (elem.)abel. A of maximal order in $T\rangle$.

Notice $J(T)$ is weakly closed in T.

Exer 3.1: Compute $J(T)$ for some small T (D_8, Q_8, ...)

(A best-case scenario:) *Assume* that $J(T) \leq C_H(V)$.

Then $J(T) \leq$ some Sylow "P" of $C_H(V)$.

And take "W" $:= J(T) = J(P)$; so this is w.cl. also in P.

Now earlier Frattini form FA gives **Thompson Factorization**:

Assume $J(T) \leq C_H(V)$ (for V, Z, W in std-sit above).

Then $\quad H = C_H\big(\ \Omega_1\big(Z(T)\big)\ \big)^{16} \cdot N_H\big(J(T)\big) \quad . \qquad$ (TF)

When *must* we have $J(T) \leq C_H(V)$—hence (TF)?

Here is one situation observed by Thompson [Tho66]:

(TF) holds for p-solvable H—except possibly
when $p = 2$ or 3, with $SL_2(p)$ involved in H.

---

[16] Ω_1 is subgp generated by elements of order exactly p

However for the general situation, methods of Thompson (cf. [GLS96, 26.12.ii]) indicate roughly that:

If *all* the locals over T satisfy (TF),

then we should expect a strongly p-embedded subgroup.

(That is, a "narrow" case—exhibited e.g. by Lie-rank 1.)[17]

This suggests that in "wider" simple groups,

we will sometimes have to deal with local subgroups

in which the desired (TF) does *not* hold...

That is, *Failure* of Factorization—abbreviated "FF".

Thompson began a line of research which led

to a useful explicit description of the FF situation...

[17] Recall (lec1): unique maxl over T if *one* node in Dynkin diagram.

§2: Failure of Thompson Factorization

Assume (TF) fails. Then $J(T) \not\leq C_H(V)$.

So: some maxl-rank $A \not\leq C_H(V)$.

(In particular, $\overline{A} := A/C_A(V) > 1$.)

Now $V \cdot C_A(V)$ is also el.ab.; so use "maxl rank" to see:

$$|A| \geq |V \cdot C_A(V)| = \frac{|V||C_A(V)|}{|C_{V\cap A}(V)|}.$$

And $C_{V\cap A}(V) \leq V \cap A \leq C_V(A)$, as A is abelian; so:

$$\frac{|A|}{|C_A(V)|} \geq \frac{|V|}{|C_V(A)|}.$$

This condition defines $\overline{A}$ as an "FF-offender" on V.

(Roughly: the codim. of $C_V(A)$ is smallish—w.r.t. $|\overline{A}|$;
so that $\overline{A}$ must centralize "much" of V ...)

There are various familiar examples of such $(V,\overline{A})$; e.g.:

(a) In V of dimension n, take a *transvection*:

$\overline{A}$ of rank 1, centralizing an $(n-1)$-subspace;

(b) Or, take any *maximal* unipotent radical of $GL(V)$:

$$\overline{A} := \overline{U}_k = \left(\begin{array}{c|c} I_k & 0 \\ \hline * & I_{n-k} \end{array}\right) \qquad \text{(UnipRad)}$$

This has rank $k(n-k) \geq (n-k) =$ codim of $C_V(\overline{A})$.

Exer 3.2: Test for FF in some small (group,module)-cases; e.g. $\overline{A} \leq \overline{H} = L_2(4)$, on natl V—over $\mathbb{F}_2$. (NOT $\Omega_4^-(2)$-mod!)

The actions in (a)(b) are even *quadratic*. (i.e. $[V, \overline{A}, \overline{A}] = 1$)

Indeed, the **Thompson Replacement Thm** [Tho69] shows:

Every offender *contains* a quadratic offender. (QFF)

In (b), $H := V \cdot GL(V)$ is a p-local of larger $GL(V_{n+1})$.

Further, FF-offenders like $\overline{A}$ *generate* $SL(V)$ (most of $\overline{H}$).

Indeed: Since the FF-offender restriction is so strong:

Can we *find* all $(\overline{H}, V)$, with (known) $\overline{H}$, gen. by such $\overline{A}$?

p-**solvable FF?** The exceptions possible in Thompson's *p*-solv were determined precisely by Glauberman ([AS04a, B.2.16]):

For *p*-solvable FF, $p = 2$ or 3—and H is direct product of terms $V_i \cdot L_i$; with V_i the natural module for $L_i \cong SL_2(p)$.

FF for more general $\overline{H}$? (Continue $F^*(H) = O_p(H)$ etc):
Idea—reduce to *components* $\overline{L}$ of $\overline{H} := H/C_H(V)$;
i.e., take $\overline{H}$ to be quasisimple $\overline{L}$.
Cooperstein-Mason (cf. [Coo78]) listed the pairs $(V, \overline{L})$ ...
...but alas, without proofs.
A full proof was given by Guralnick-Malle [GM02]; one has:
(Quasisimple FF-list:) $\overline{L}/Z(\overline{L})$ is either:
Lie type in char p, or alternating ($p \leq 3$); with V "small".[18]

This FF-list is much used in the CFSG; e.g. $C(G, T)$
Theorem below; and abstract minl parabolics (lec4).

So we begin some development in the direction of applications:

[18] We'll explore such representation-theoretic concepts in later lec5.

§3: Pushing-up (FF-modules in Aschbacher blocks)

(Take $p = 2$ in this section; many ideas are OK for general p.)
Much analysis in the CFSG[19] uses the *Thompson strategy*:
Given a "first" (maximal) 2-local M containing Sylow_2 T,
if possible find a *second* local H over T—with H *not* in M.
(Can then e.g. consider the *larger* group $\langle M, H \rangle$ ... etc)

But what if it is *not* possible to find such an H?
That is, when T lies in a *unique* maxl 2-local ("M").
The Strongly Embedded Theorem handles a special case...[20]
... but desirable to determine the possibilities in general.

In particular, $N_G(T)$ then lies in that unique maximal M;
as does any local above $N_G(T)$.
Here is one approach to some natural locals over $N_G(T)$:

---

[19] i.e. in orginal CFSG; but even in *any* known approach to CFSG?
[20] including the "expected" conclusions having Lie rank 1

For any C *characteristic*[21] in T, we have $N_G(T) \leq N_G(C)$.
Ideally, get $<$; i.e. "push up" to larger $N_G(C)$.

But if *not* ? I.e. $N_G(C) = N_G(T)$ for all such C ? Suggests:
Set $C(G, T) := \langle N_G(C) : 1 < C \text{ char } T \rangle$; consider situation:

$$C(G, T) \leq M < G, \qquad \text{(CPU)}$$

Note: if T in unique maxl M, all $N_G(C) \leq M$; get (CPU).

Suppose first we have (CPU) in a 2-*local* subgroup H over T:
that is, $C(H, T) \leq M_H < H$.
Then (CPU) is "failure" to push up in H.
Furthermore, H then fails factorization (FF);[22] roughly:

- the factors in (TF) lie in $C(H, T)$
- so their product also $\leq C(H, T) \leq M_H$, which is $< H$.

Can we characterize this (CPU)-situation?
(I.e., specify possible "obstructions" to pushing-up?)

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[21] Meaning: any automorphism of T must normalize C
[22] See e.g. [AS04a, C.1.26] for details.

Aschbacher's **Local** $C(G,T)$-**Theorem** [ALSS11, B.7.3] says:
If $F^*(H) = O_2(H)$ and $C(H,T) < H$,
then $H = C(H,T)L_1 \cdots L_t$, for "χ-blocks" L_i;
where blocks arise from only a small subset of the FF-List:
An (Aschbacher) block has $\overline{L}_i \cong L_2(2^m)$, or A_m (m odd);
on V with a *unique* nontrivial irreducible FF-section.

This led to the **Global** $C(G,T)$-**Theorem** [ALSS11, B.7.8]:
If G is simple of char. 2 type, with $C(G,T) < G$,
then G is (in a list; notably Lie rank 2, where $\exists$ graph aut).
Exer 3.3: Find blocks in locals of $L_3(2^m)$; $Sp_4(2^m)$. ($L_3(2)$:
$C(G,T) = T$ as rk-1 unip rads 2^2 *not* char; 2^2L_2 is block.)

After this, we may assume T lies in at least 2 maximal locals;
can use Thompson strategy—locals M, and $H \not\leq M$, over T.

Above, rk-1 $L_2(2^m)$ appears in some blocks of rk-2 groups;
the $C(G,T)$ situation could be called rank-1 pushing up.
For we can generalize that situation e.g. to higher rank:

Pushing up using $C(G,R)$ with $R < T$

We generalize from T by taking some $R \leq T$;
but assuming further that $R = O_2\big(N_G(R)\big)$.[23]

Then (CPU) takes the form $C(G,R) \leq M < G$;
where we also assume that R is Sylow in $\langle R^M \rangle$.
(This holds for example if we have $M = N_G(R)$.)

There are results describing (CPU) for various $R < T$;
again locals H via blocks—which are now larger than before:

- bigger groups $\overline{L}$; acting on:
- modules V satisfying conditions weaker than (FF).

Example 1: Meierfrankenfeld and Stellmacher develop results on **rank-2 pushing up** (e.g. [AS04a, C.1.32]).

Roughly: R is a rank-2 unip-radical (in T from Lie-rank 3);
and the more general blocks arising here
can be used in various ways in the CFSG:

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[23] Such R are called 2-*radical*. (Important e.g. for topological view...)

Example 2: A quasithin (CPU) use—[AS04c, 8.1.1]:
How can we rule out the QT local: $\overline{L} \cong M_{23}$ on $V \cong 2^{11}$?
(The "shadow" of Fi_{23}—which can't be our G, as *not* QT.)

By QT "theory": $LT \leq$ *unique* maxl 2-local $M = N_G(L)$.
Then for $R := O_2(LT)$, LT normalizes any C char R;
hence $C(G, R) \leq M < G$—i.e., we have (CPU) for R.
How used? Note there is $x \in V$ with $C_{\overline{L}}(x) \cong M_{22}$.

Assume $C_G(x) \not\leq M$. Then (via "big" M_{22}) it inherits (CPU):
$$C\big(C_G(x), R\big) \leq C_M(x) < C_G(x).$$
So $C_G(x)$ should be in a corresponding global-(CPU) list—
—but proof of [AS04c, C.2.8] shows such groups aren't QT.
(Notably: $C_{Fi_{23}}(x) \cong \langle x \rangle Fi_{22}$ is not QT.)

Thus we conclude that $C_G(x) \leq M$.

(And since that *differs* from the situation $C_G(x) \not\leq M$ in Fi_{23},
we have a first step, toward ruling out L under QT hyp.
To be continued...)

§4: Weak-closure factorizations—other "W"

(Ideas from Thompson—further development by Aschbacher.)

To motivate: If some conjugate V^g lies in T,
 we might *expect* $V^g \leq$ some maxl-rk A;
 if always so, then weak closure of V falls into $J(T)$.

So: use w.cl.(V) (maybe $< J(T)$) as "W" in factorizations?

We follow Aschbacher's axiomatization; for $i < m(A)$, set:
 $W_i := \langle A : A \leq T \cap V^g, m(V^g/A) = i \rangle$;[24]
 $C_i := C_T(W_i)$.

Then use some of these for "W, Z", in analogues of (FA)
Below, I'll blur most details; but see e.g. [ALSS11, B.8.5].

In Thompson strategy with $H \not\leq M$, usually V is M-module.
And we may want to factorize H, with module U—
 —when we can get the hypothesis $1 < W_i \leq C_H(U)$.

[24] Notice for $i := 0$: W_0 is the usual weak closure of V in T.

This also requires estimating lower bounds
on certain "parameters" $m, r, s, a, n(H)$...
Roughly: their defns are abstracted from the "usual case";
where $\overline{M} := M/C_M(V)$ is of Lie-type in char 2,
and V is described via 2-modular repn theory (lec5) of $\overline{M}$.
(Similarly for $\hat{H} := H/C_H(U)$ on U.)
E.g. $s = \min(r, m)$, where:
r is global lower bd—corank $A \leq V$ with $C_G(A) \not\leq M$;
m is M-local lower bound on corank of C_V(inv of $\overline{M}$).
These are related to ranks of root-spaces on V,
as well as to ranks of root-groups in $\overline{M}$.
Similarly, a is roughly the rank of a root-space on V;
and $n(H)$ is roughly the rank of a root-group in $\hat{H}$.

And now to *roughly* state Aschbacher's [Asc81b, 6.11.2]:[25]
Assume $W_i > 1$, for some i with $0 \leq i < s - a$. Then:

$$H = C_H(C_{i+n(H)})N_H(W_i). \quad \text{(WC)}$$

Here are two CFSG areas where (WC)-methods were applied:

[25] Cf. also [ALSS11, B.8.5]

Uniqueness Case: (Final contradiction in CFSG)
Aschbacher [Asc83b] eliminates:
"almost strongly p-embedded" maxl 2-local $M \geq T$. How?

Many subcases are handled via the following overall approach:
Recall Thompson strategy: further local H, with $T \leq H \not\leq M$.
(Ideally) use (WC) to get $H = H_1 H_2$.
Then—use uniqueness properties of M...methods like (CPU);
to (ideally) force $H_i \leq M$.
Then $H = H_1 H_2 \leq M$—contradicting choice of $H \not\leq M$!

Quasithin: Also Thompson strategy: maxl M on V; $H \not\leq M$.
Set $w :=$ smallest i with $W_i \not\leq C_G(V)$—"where (WC) fails".

Get natural *upper* bounds on w; e.g. [AS04a, E.3.39]:[26]
(Certain technical hypotheses) $\implies$ $w \leq n'(\overline{M})$,
where $\mathbb{F}_{2^{n'}}$ gen by largest odd-order sgp permuting with T.

So—enlarge *lower* bounds on w; notably [AS04a, E.3.29]:
If V is not an FF-module, or we have $w > 0$, then:

$$w \geq r - m_2(\overline{M}). \quad \text{(FWCI)}$$

Example 2—continued: Recall $V = 2^{11}$ under $\overline{L} \cong M_{23}$.
A 3-element permutes with Sylow_2 of M_{23}; so $w \leq n'(\overline{M}) = 2$.
However: we earlier got $C_G(x) \leq M$—here, forcing $r \geq 7$.
(*Unlike* shadow F_{23}, where $r = 6$.) So by (FWCI):

$$w \geq 7 - m_2(M_{23}) = 7 - 4 = 3.$$

This contradicts $w \leq 2$ above; so eliminates case of such $\overline{L}$!

We'll PAUSE here; before moving on to more recent work...

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[26] the proof uses uniqueness methods, e.g. (CPU)

Lecture 3a: Oliver's Conjecture on $J(T)$ (for odd p)

(version of 25sept2015)

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Venice Summer School
7-18 September 2015

Overview of the talk:

Introduction: the Martino-Priddy conjecture
1: Oliver's original proof—using the CFSG
2: Oliver's Conjecture on $J(T)$ for p odd (would avoid CFSG)

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

Introduction: the Martino-Priddy conjecture

We had earlier mentioned the fusion-systems approach;
for context, we now provide (just a little) background:

For G with p-Sylow T, the *p-fusion* means the data:
$\{$ all G-conjugacies among subgroups of T $\}$.
(*Not* limited to conjugacies induced within just $N_G(T)$!)
Exer 3.4: Find fusion for $G = A_6$, inside $T \cong D_8$; etc...

This info has long been important in various ways; eg:

- in CFSG etc: Alperin Fusion Thm; "recognition" thms; ...
- for alg topol, in computing group cohomology $H^*(G)$ ...

Classical topol approach to cohom—via classifying space BG;
indeed for p-part $H^*(G)_p$, use p-completion $BG_p^{\wedge}$.
Modern simplicial-topology gets these spaces from *categories*;
can we approach cohom via fusion—using such a category?
Puig (1980s) pioneered; for a modern survey, see [BLO03].
Below we give just a very rough introduction:

A *fusion system* $\mathcal{F}$ on a p-group T is a category including:
Objects: the subgroups (P, Q, ...) of T;
Morphisms: suitable injections $P \to Q$ (including T-inner);
with axioms (roughly replacing some Sylow theory). Examples?

- (standard) For T Sylow in a group G, get $\mathcal{F}_{T,G}$: by using the G-conjugacies as the morphisms.
- ("exotic") eg *Benson-Solomon* system: *not* from finite G. (Inf.ext. of Spin_7-gp; rel: Dwyer-Wilkerson space $BDI(4)$.)

Here is one way of precisely expressing that, for groups, fusion systems should be equivalent to cohomology;
namely the **Martino-Priddy conjecture** [Oli04, Thm B]:
For groups $G \geq T$ and $G^* \geq T^*$,

$$BG_p^\wedge \simeq_{\text{homot.eq.}} (BG^*)_p^\wedge \quad \Leftrightarrow \quad \mathcal{F}_{T,G} \equiv_{\text{categ.eq.}} \mathcal{F}_{T^*,G^*}$$

In fact Martino and Priddy proved $\Rightarrow$ ...
But $\Leftarrow$ was harder—because of certain $\lim^2$-obstructions.

§1: Oliver's proof using the CFSG

Oliver proved $\Leftarrow$ in [Oli04, Oli06] (for p odd, $p = 2$).
He uses CFSG;[27] we will sample some of those arguments.

(Digression:) a quick (no-defns!) overview of [Oli04]—p odd:
Show vanishing of higher limits $\lim^{i>0}$
of center-functor $\mathcal{Z}$ defined on G-orbit category.
This removes obstruction to $\exists!$ of "linking" system—
—for building from $\mathcal{F}_{T,G}$ to classifying space. (End Digr.)

Oliver reduces to simple G at 4.1 of [Oli04]. Now need:
a Q which is: centric; w.cl. under $\mathrm{Aut}(G)$; and $\leq X(T)$;
where $X(T)$ is a char subgroup of T (defn in a moment),
and centric means $Z(Q)$ is Sylow in $C_G(Q)$. Note:
(*) If $X(T) = T$, then T as "Q" works; so done...

So, let's sample properties of $X(T)$ from Sec 3 of [Oli04]:

[27] Chermak's later proof [Che13] makes "milder" use of the CFSG; and recently, Glauberman-Lynd gave a CFSG-free proof.

(Defn:) $X(T)$ is the (unique maxl) final member "R_n", among chains of subgroups normal in T:

$$1 = R_0 \leq R_1 \leq \cdots \leq R_n \leq T,$$

which satisfy $[\ \Omega_1\big(C_T(R_{i-1})\big)\ ,\ R_i\ ;\ p-1] = 1$.
(For $p = 2$, get $X(T) = C_T\big(\Omega_1(T)\big)$—not useful for [Oli06].)

Among properties are [Oli04, 3.2,3.10]:

(A) Abelian normal subgroups lie in $X(T)$ (which is centric).
($+$) If $\Omega_1(\ Z\big(X(T)\big)\ )$ has rk $< p$, then done via (*).

Exer 3.5: Find $X(T)$ for some smallish (nonabelian) T...

Further sufficient conditions appear in [Oli04, 3.7]:

(a) OK if $X(T) \geq J(T)$ (use $J(T)C_T\big(J(T)\big)$ as "Q");
(b) OK if $J(T)$ is unique el-ab of maxl-rank (via (A),(a));
(c) OK if $T/X(T)$ is abelian.

(Indeed [Oli04, Conj 3.9] asks if (a) is true—indep of CFSG?)

We sketch Oliver's check of simple G [Oli04, 4.2–4.4]:

Many cases go via (b)—unique elem abel of maximal rank:

- alternating: Recall lec1a—unique of maxl rank $\lfloor \frac{n}{p} \rfloor$.
- Lie($r \neq p$): unique maxl via *diagonal* (incl *non*-split).
  (Also works for some of the remaining cases below.)

Exer 3.6: Study non-split in $L_4(2)$ for $p = 3, 5, 7$...

For Lie(p): most cases have $X(T) = T$, so done by (*);[28]
otherwise—a unipotent radical in $X(T)$ works as "Q".

Finally for sporadic:

- For $p > 3$, $m_p(G) < p$—so done by $(+)$.
- For $p = 3$, various methods can be used...
  E.g., (at least) 11 cases similarly have $m_3(G) < 3$...$(+)$
  Then (at least) 8 more have (c): $T/X(T)$ abelian.
  Remaining cases: with some work, via (b) or (*).

[28] This remark after [Oli04, 4.3] can be used to replace the incorrect statement $U_J \cap U_{J'} = U_{J \cup J'}$ there.

§2: Oliver's conjecture on $J(T)$ for p odd

We saw that Oliver conjectured in [Oli04, Conj 3.9] that (a) above should in fact always hold:

Conj: For T a p-group (p odd), $J(T) \leq X(T)$.

(Would give (another) CFSG-free pf of odd Martino-Priddy.)

In a minimal counterexample to this J-Conjecture,

- $X(T)$ is elementary abelian (so we'll call it V).
- Further V is an FF-module under $\overline{T} := T/C_T(V)$.

Set $n := \dim(V)$. Various results show "not too small"; e.g.:

- Oliver [Oli04, 3.10]: $n \geq p$
- [GHM10]: $\mathrm{cl}(\overline{T}) \geq 5$; (Lynd:) $\mathrm{cl}(\overline{T}) \geq \log_2(p-2)+2$

Variant (e.g. [GHM11, 1.4]): Recall $\exists$ quadratic offenders (QFF)

Q-Conjecture: If V is an FF-module for odd p-group $\overline{T}$, then $\Omega_1\big(Z(\overline{T})\big)$ contains a quadratic element.

Eg: a ctrex $\overline{T}$ has no quadratic *normal* subgroup.

(also: "no transvections" seems implicit in the literature...)

These last remarks suggest connections with earlier-today:
Embed ctrex $\overline{T}$ in a full unipotent group $\overline{U}$ of $GL(V)$.
Then for maxl (quadratic!) $\overline{U}_k \trianglelefteq \overline{U}$ of earlier (UnipRad)

$$\overline{T} \cap \overline{U}_k = 1.$$

Eg: A transvection $\overline{t}$ lies in some $\overline{U}_k$ ($[V, \overline{t}] \leq V_k$); so:

$\overline{T}$ contains no transvections.

Also: map of $\overline{T}$ into $\overline{U}/\overline{U}_k$ must be faithful; so use $k = \lfloor \frac{n}{2} \rfloor$:

$$\mathrm{cl}(\overline{T}) \leq \tfrac{n}{2}, \text{ and}$$
$$\nu_p(\ |\overline{T}|\ )\ \leq \tfrac{1}{2}\tbinom{n}{2}.$$

(Ambitious exercise: prove, or disprove, the Conjecture!)

So—there are still mysteries, about some details of FF.

To conclude: Thompson factorization—
—the story will be continuing long after us...

THANKS!

Lecture 4: Recognition theorems for simple groups

(version of 25sept2015)

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Overview of the talk:

Introduction: Finishing classification problems
0: Background—uniqueness and recognition
1: Recognizing alternating groups
2: Recognizing Lie type groups
3: Recognizing sporadic groups

longer handout: `www.math.uic.edu/∼smiths/talkv.pdf`

Introduction: Finishing classification problems

In lec3(Ex 2), we sketched the elimination of 2-local $2^{11}M_{23}$;
i.e. of the "shadow" of Fi_{23}—not arising in QT classif.
On the other hand—in classification problems in general:
how do we finish cases, that *do* arise in desired concl ?
For this, we use suitable "recognition theorems"; showing:
(sufficient local-info) $\Rightarrow$ (identification of global-G) .

Even in problems not quoting CFSG, using recognition thms
is much in the spirit of "application" of the CFSG-list...

Some general discussions of recognition theorems include:
Ch 3 of [Gor82], and Sec A.5 of [ALSS11].
There are *many* recognition theorems in the literature;
with titles like "A characterization of ...some group(s)..."
Today we will sample some of them.

§0: Background—uniqueness and recognition

Implicit in the CFSG are the properties of
existence and uniqueness for the groups in the list.

Existence: At outset, already know many (most?) groups;
they exist via obvious constrs: permutations, matrices ...
But subtleties do arise—in seeking completeness of CFSG-list:
Ex: Relating many groups via the unified theory of Lie type
led also to discov/constr of new groups of Suzuki, Ree ...
Indeed, proofs of intermediate thms could lead to new groups;
namely: a configuration that couldn't be eliminated
might be from new group. (So: add concl; re-start proof!)
A famous ex: Fischer's groups (in trying to characterize S_n).
Also: Fischer, Griess found (1973) "evidence" for Monster...
...and soon Griess gave his celebrated construction (1980).

(How avoid missing some? As with any mathematical proof:
write with care; get it *read* with care; repeat as nec...)

Uniqueness: Each G should be defined uniquely (up to $\cong$).
(Not always obvious: E.g., Ree-group subtleties;
resolved by Thompson-Bombieri [Gor82, 3.38].)
Note: the usual form of the CFSG-list implicitly assumes
some "small" cases of $\cong$ across types [ALSS11, p 261]
(e.g. $A_5 \cong L_2(4) \cong L_2(5)$, as we had seen in Exer 1.5).

Of course: "recognition theorems" as introduced earlier
are basically just a variation of uniqueness of a given G.
(In effect—they consider various sets of properties,
which are sufficient to uniquely identify G.)

So: what kinds of information should be sufficient—
—that is, as hypothesis in recognition theorems?
There are *many* approaches to recognition...
...but as the results tend to be rather technical,
we'll give only an overall description of some common types.

I'll make a rough distinction in the next few pages:
First consider *final* (or fundamental) identification of G;
i.e. properties reasonably close to the defn of G.

A group-element approach: via a presentation for G:
i.e. using generators and relations.
This is one standard approach to A_n;
and often for Lie type (cf. Curtis-Tits presentation below).
However, these can be touchy; and not always very instructive.

Instead, it can be more natural to use a group *action*—
—on some "object" distinctive enough to characterize G.
Here the object might be only a set;
typically with a highly-transitive action.
Or, a graph; or higher-dim geometry; or lattice; or module ...
We'll see various examples below.

Next—in contrast to "final" recognition—we can turn
to more varied possible *initial* hypotheses;
which, with work, can lead to that final identification.

One standard hypothesis: the *involution centralizer* $C_G(t)$.
Ex: standard-form problems (lec2) are in this spirit.
(But "careful": $2^{1+6}L_3(2)$ in 3 groups—$L_5(2)$, M_{24}, He.)

A related hypothesis is structure of *the Sylow 2-subgroup* T ...
E.g. T *abelian* (Walter, Bender e.g. [Gor82, 4.126]):

- $L_2(2^a)$—uniquely for $a \geq 4$; plus
- $a = 2$: $L_2(q), q \equiv 3, 5 \pmod 8$; $a = 3$: J_1, Ree groups.

The Small Odd Case $m_2(G) \leq 2$ (lec2) used similar classifs
for T dihedral, semi-dihedral, wreathed [ALSS11, 1.4.6].

(The above are examples of recognition via *local subgroups*.)

Another related hypothesis is the *group order* $|G|$.
This order was often computed via the
Thompson Order Formula [Gor82, 2.43]:
For G with 2 conj classes of involutions (say t, u),

$$|G| = a(u)|C_G(t)| + a(t)|C_G(u)|;$$

for $a(v)$ the number of products xy from the 2 classes
with $v \in \langle xy \rangle$. (centralizers+fusion (lec3a) suffices...)
Exer 4.1: Use formula for order of S_4, S_5; even A_8 ?
(See notes at `www.math.uic.edu/~smiths/exer41.pdf`)

A further variant hypothesis is the *character table* of G.
(Implicitly includes group- and centralizer-orders.)

We now turn to more specialized recognition,
for the types of simple groups in the CFSG-list.

§1: Recognizing alternating groups

The group A_n is arguably fairly easy to recognize;
at least, given the natural perm repn of degree n.
Some pre-CFSG results on 2-trans appear in [Gor82, Sec 3.2].
(Indeed *assuming* the CFSG, the 2-trans list of lec1a
shows that 6-transitivity suffices to recognize A_n.)

Recognizing A_n just from local information can be harder...

But often a standard presentation [Gor82, 3.42] suffices:
based on properties of involutions in A_{n-2} and a 3-cycle.

Fischer approached almost-simple S_n, via similar properties;
using (generation by a class of) *3-transpositions*:
involutions with $|xy| = 1, 2, 3$ for x, y in the class.
Here some other almost-simple groups arise [ALSS11, A.6.3]:
some classical/$\mathbb{F}_2$; some orthogonal/$\mathbb{F}_3$; Fi_{22}, Fi_{23}, Fi_{24}.
Exer 4.2: Find 3-transps in classical cases. (transv,refl)
(This viewpoint led to other useful classifications...below)

§2: Recognizing Lie type groups

Perhaps the most natural recognition of (most) Lie-type G is as the autom-group of its *Tits building* [Car89, Sec 15.5]— —a simpl-complex "axiomatized" by the Dynkin diagram.

Crucial for "topological" uniqueness is Tits' thm [Ron89, 4.3] that for rank ≥ 3, a finite bldg is *simply connected*. (So: comes "uniquely" from Lie type group [Gor82, 3.12].)

Correspondingly, at the level of the elements of the group G, we get the **Curtis-Tits Presentation** [ALSS11, A.5.1]:[29]
(Recall the Lie algebra $\rightarrow$ root-subgroups setup from lec1)
For (untwisted) G with simple system Π of rank ≥ 3,
the generators $\langle U_{\pm i} \rangle \cong SL_2(q)$ for $i \in \Pi$,
and relations $\langle U_{\pm J} \rangle$ for rank-2 $J \subset \Pi$ "as in the diagram",
give a presentation for G (univ: with Schur mult).
Exer 4.3: Give the rank-2 gps, for pairs in a few diagrams.

Applied? Frequently; e.g. for recog in Standard-Type (lec2).

---

[29]Using earlier *Steinberg relations*.

Twisted groups are a little more complicated;
a unitary-groups analogue of Curtis-Tits was given by Phan.
(Revised: Bennett-Shpectorov [ALSS11, A.5.2].)

So, what about Lie ranks ≤ 2?

The cases of rank 1 were mostly recognized
using (pre-CFSG) 2-transitive results [Gor82, Sec 3.2].

For rank 2, several approaches have been used, including:
early: *split BN-pairs of rank 2*—Fong-Seitz [FS73];
recent: *Moufang generalized polygons*—Tits-Weiss [TW02].
(See [AS04b, Sec F.4] for "amalgam+centralizer" approach.)

We briefly mention some more specialized characterizations:

Generalizing beyond Fischer's 3-transposition condition, Timmesfeld [Tim73] gave more groups over $\mathbb{F}_2$ via (generation by a class of) $\{3,4\}^+$-*transpositions*: now $|xy| = 1, 2, 3$; or 4, with $(xy)^2$ in the class.
Applied? e.g. for recognition in the $GF(2)$-Type case (lec2).
(Groups over larger $\mathbb{F}_{2^a}$ were treated by Timmesfeld [Tim75] via *root involutions*—$\{\text{odd}, 4\}^+$-transpositions.)

An early result of McLaughlin [ALSS11, A.6.1] classified groups generated by *transvections* (lec3a) on irred $V/\mathbb{F}_2$: namely $SL(V)$, $Sp(V)$, $SO^{\pm}(V)$, S_{n+1}, S_{n+2}.
Exer 4.4: Exhbit transvections in some small cases.

Aschbacher's Classical Involution Theorem [ALSS11, 1.7.5], and viewpoint of "fundamental $SL_2(q)$s" in odd Lie-type, were important for Standard-Form analysis (lec2).

§3: Recognizing sporadic groups

See [Gor82, 3.50] for various involution-centralizer results.

A weakening of 2-transitivity is a *rank-3* permutation group:
where G_α has *two* orbits on points $\neq \alpha$. ("G-suborbits")
Ex: For classical G of V with a form, trans on the isotropic v,
from Witt's Lemma (lec1) we get roughly:
G_v trans on (all other $w \perp v$) and (all $x \not\perp v$).
Exer 4.5: Give suborbits for $O_4^-(2) \cong S_5$ on 10;
and for $Sp_4(2) \cong S_6$ on 15; cf. [Smi11, Secs 2.1–2.2].

But also: some sporadics...
e.g. J_2, HS, McL, Suz, Ru (+ Fischer groups)
were found (and characterized) via rank-3 repns.
See e.g. [Gor82, Sec 2.6] for further details.

Aschbacher in Part III of [Asc94] develops (and applies)
a fairly uniform approach to recognition for sporadics,
in the context of simple connectivity of suitable graphs.
We'll PAUSE here; before moving on to "applications"...

Lecture 4a: Recognizing some quasithin groups

(version of 25sept2015)

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Venice Summer School
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Overview of the talk:

Introduction: Some "quasithin local theory"

1: Recognizing Rank-2 Lie type groups

2: Recognizing the Rudvalis group Ru

longer handout: `www.math.uic.edu/∼smiths/talkv.pdf`

Introduction: Some "quasithin local theory"

To set the stage for recognition of QT-groups, we first give some background, on the overall structure of [AS04d]:

In a lengthy classification problem,
(e.g. subproblems within CFSG, such as quasithin)
often one develops a "theory" of subgroups—especially locals.
This is carried out for QT particularly in Ch 1 of [AS04d].
(And we'll say more on that, in a moment.)

But first, we recall some considerations much as in earlier lec2.
Ch 2 of [AS04d] gives a "$C(G,T)$-Thm for quasithin":
the QT-subcase with 2-Sylow T in a unique maxl 2-local.
Thus *after* Ch 2, we can adopt the Thompson strategy:
where M is a maximal 2-local over T,
and H is another 2-local over T, with H not lying in M.
In fact, we normally take H *minimal* with this property.

We return to "local theory", to describe H and M.

The case of H is fairly quick to sketch:
Minimal choice gives "$H \in \mathcal{H}_*(T, M)$" [AS04d, 3.0.1]:
by [AS04d, 3.3.2(4)], H is an "abstract minimal parabolic".
(In the sense of McBride [AS04b, B.6.1].)
Then $H/O_2(H)$ is described in [AS04b, E.2.2]:
mostly of rank-1 Lie type in char 2 (recall lec1: L_2, U_3, Sz).
(Indeed H often has FF on an internal module; and then
$H/O_2(H)$ is (lec3:FF-list) usually $L_2(2^m)$ [AS04b, E.2.2].)
Such H does resemble a minimal parabolic in even Lie-type...

For M chosen maximal, there will be more possibilities...
In fact M arises as $N_G(L)$, for a suitable "$\mathcal{C}$-component" L:
simple mod solvable, described in [AS04d, Sec 1.2].[30]
The maximality is expressed via "$L \in \mathcal{L}_f^*(G, T)$";
where f indicates faithful on suitable module V in $O_2(L)$.

[30] In QT local-theory, these have *some* properties of components...

"Usually" we have the Fundamental Setup of [AS04d, 3.2.1]; and subsequent results in Sec 3.2 determine the case-list for $\overline{L} := L/O_2(L)$ and the internal module V.

Typically $\overline{L}$ is even Lie-type of rank 1 or 2, with V "small".
So: must treat cases $(\overline{L}, V)$ in M—versus possible $H \not\leq M$.

By [AS04d, 3.3.2(1)], M is the *unique* maxl 2-local over LT.
So for $R := O_2(LT)$, $C(G, R) \leq M < G$.
This (CPU)-condition enables pushing-up argts—recall lec3.

The "majority" of cases are treated in the next few chapters:

- Most conclusion-groups are even Lie-type of rank 2; these are treated in Ch 5 (as sketched starting next page).
- Most of the *un*likely cases have V a non-FF module; these "shadows" are eliminated in Ch 7–9. (We had outlined case $\overline{L} = M_{23}$ on $V = 2^{11}$ in lec3:Ex2.)

Thus Chapters 10–16 of [AS04d] are devoted to treating comparatively few cases of $(\overline{L}, V)$.
(But those small cases are disproportionately complicated...)

§1: Recognizing Rank-2 Lie type groups

We outline Ch 5 of [AS04d]. ("main" conclusion gps in QT)

Hypothesis 5.0.1 is: $\overline{L} \cong L_2(2^n)(n \geq 2)$. (Various possible V.)
(Cases $n = 1$ for L, H mostly postponed to later chapters.)

As $H/O_2(H)$ is also (usually) a rank-1 group,
both L and H "look like" parabolics in rank-2 Lie-type.
E.g., $T \cap L$ should be like a full unipotent group;
and $N_G(T \cap L)$ should be like a Borel subgroup.
But this requires nailing down many further details ...

And much initial detail for V, H is refined in Sec 5.1.
E.g., note "Borel" $N_L(T \cap L)$ is solvable;
so a Hall $2'$-subgp plays the role of (most of) Cartan group.

Then Sec 5.2 identifies possible corresponding "amalgams"...
Namely by verifying [AS04b, F.1.1]:
a version of the "weak BN-pairs of rank 2" determined
by Delgado-Goldschmidt -Stellmacher [DGS85]. Roughly:
Input: (2 parabs + intersection)—known *only* mod O_2;
Output: determines the full possible Sylow 2-group(!).
This does not yet identify our G generated by the parabs.
(Just gives an *infinite* group that "covers" G.)

The possible amalgams are those for rank-2 even Lie-type...
(and one exception: the $L_3(4)$-amalgam from two $L_2(4)$'s
has an extension, where one $L_2(4) \cong A_5$ extends to A_7;
Here we identify (quasithin conclusion group!) G as M_{23}:
with recognition via the method of [Asc94, 37.10].)

Section 5.3 moves from amalgams to corresponding groups:
Initial lemmas include structure of $C_G(t)$ for 2-central t.
This, together with the amalgam generating G,
is sufficient to recognize G—using [AS04b, F.4.31].
(Roughly: relations from the finite group $C_G(t)$ suffice to "collapse" the infinite cover down to the desired G.)

So at the end of Chapter 5 in [AS04d],
the infinite families in the conclusion have now arisen.
Alas, the analysis of the remaining "small" cases
(which may or may not lead to conclusion-groups)
will occupy another 11 chapters.

In fact the last case of the Fundamental Setup to be treated
is $\overline{L} \cong L_3(2)$ in Ch 14—with V natural (after 12.4.2).
We now turn to the very last subcase of that $L_3(2)$-analysis:

§**2: Recognizing the Rudvalis group** Ru

In Sec 14.7 of [AS04d], we have $\overline{L} \cong L_3(2)$ on natural V.
Secs 14.3 and 14.4 had shown that for 2-central z,
$U := \langle V^{C_G(z)} \rangle$ nonabelian leads to $G \cong HS$ or $G_2(3)$.
(Various recognitions discussed at [AS04b, I.4.8, I.4.5]; including rank-3, inv-centralizer, weak BN-pair ...)
So Sec 14.7 begins with U abelian;
and taking $C_G(z)$ for "H", gets $H/O_2(H) \cong S_5$;
with specific action on sections of U of dimensions $1, 4, 6$.
Then shows $O_2(H) = U$; and $O_2(L)/V$ the 8-dim adjoint-mod.
(So that H and L are just as in Ru.)
These are hypothesis for [AS04b, J.1.1]—recognizing G as Ru.
(That result in turns uses recognition by rank-3 perm repn.)

THANKS!

Lecture 5: Representation theory of simple groups
(version of 27sept2015)

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Venice Summer School
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Overview of the talk:
Introduction: Some standard general facts
1: Representations for alternating groups
2: Representations for Lie type groups
3: Representations for sporadic groups

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

Introduction: Some standard general facts

Simple groups are often used via (linear) *representations*.
(One reference for general repn-th is Curtis-Reiner [CR90].)

A representation is a homomorphism $G \to GL(V)$,
for V a vector space over a field F.
Typically we reduce to the case where V is *irreducible*.

("Ordinary":) If $\mathrm{char}(F) = 0$ (or $p \not| \, |G|$ for large-enough F),
then the group algebra FG is semisimple;
and repns are completely reducible (i.e., sum of irreducibles).
Their characters (traces) determine decompositions etc ...
(See e.g. Isaacs [Isa06] for basic character theory.)

("Modular":) Instead when $p = \mathrm{char}(F)$ *does* divide $|G|$,
then the *indecomposable* direct summands of FG (in blocks)
are the *projective* covers $P(I)$ of the irreducibles I.
The p-modular case is important for p-local subgroups etc ...
(See e.g. Feit [Fei82] for basic modular theory.)

For simple groups (especially in the modular case),
often suitably "small" representations are important:
e.g. $\Omega_1 Z\big(O_p(M)\big)$ for a p-local M. ("internal" module)
Indeed for many groups, the smallest-dim nontrivial module
may be considered the "natural" module V for G.

Smaller V, W lead to larger, via the tensor product $V \otimes W$;
esp. symmetric and skew-symmetric powers $S^k(V), \Lambda^k(V)$.
(E.g. often there is also an "adjoint"—related to $\Lambda^2(V)$.)

Given a group extension, Clifford's Theorem [CR90, 11.1]
describes how repns of the factors
are assembled into repns of the extension.

For more detail, we consider the types in the CFSG-list:

§1: Representations for alternating groups

Typically it is natural to first discuss repns of almost-simple S_n;
and then pass down to A_n via Clifford's Theorem above.

(Ordinary:) First naively, for the smallest irreducibles:
Perm repn P, from lin-combs of the n "points", is sum of:

- 1-dim triv submod T (from all-same-coeff vectors);
- $(n-1)$-dim *natural* irred N (from coeff-sum-0 vectors).

General theory of irreducibles for S_n is a classical topic:
combinatorics of Young diagrams—e.g. [JK81] [Sag01].

Cycle-type shows: (conj classes) $\leftrightarrow$ (partitions λ of n).
So get # irreds; even get explicit (irreds) $\leftrightarrow$ (partitions):
The irred I_λ has basis indeed by *standard Young tableaux*:
put $1 \cdots n$—*increasing* in rows/cols, of diagram from λ.
(The dimension of I_λ is given by the "hook-length formula".)

- Trivial T from $\lambda = n$: single row has only *one* incr-order.
- Natl N from $\lambda = n-1, 1$: use $2; \cdots; n$; for elt in row2.

Exer 5.1: Check; explore some other examples for small n.

(Modular:) Natural $N \cong P/I$ above—except when $p \mid n$:
for then T instead lies *in* the coeff-sum-0 submodule N,
so that $\overline{N} := N/I$ is irred—now of dim $n-2$.

Here the I_λ-theory has to be further refined.
Fewer: Now, have (p'-order classes) $\leftrightarrow$ (p'-partitions),
namely where part-sizes are not divisible by p.
The latter in turn correspond with p-regular partitions,
in which no p successive parts have the same size;
and these partitions λ will $\leftrightarrow$ the modular irreducibles:
However, the "naive" char-0 I_λ are usually reducible mod p;
a certain quotient $\overline{I}_\lambda$ is now the relevant irreducible.
The theory is still being developed; e.g., still not known are:

- the dimensions of the irreducibles $\overline{I}_\lambda$;
- the *decomposition matrix* (I_λ as sum of "smaller" $\overline{I}_\mu$).

(Connection:) Recall S_n is the Weyl group in GL_n.
The repn theory of S_n leads to some repns for GL_n ...

§2: Representations for Lie type groups

(Ordinary:) See e.g. Carter [Car93]. I'll roughly sketch some Deligne-Lusztig theory. (Cf. more recent *Lusztig induction*.)

Recall from lec1: can obtain finite Lie-type G as fixed points in alg gp $\overline{G}/\overline{\mathbb{F}_p}$, under automorphism F.

Fix an F-stable maxl torus $\overline{T}$ ("Cartan sgp") in $\overline{G}$.

(These $\overline{T} \leftrightarrow$ "F-conjugacy" classes in Weyl gp W;
roughly, $\overline{T}^F$ is (?non-split) diagonal subgroup of G.)[31]

Further fix an irred (i.e., 1-dim!) character θ of $\overline{T}^F$.

Now for prime $\ell \neq p$, define $R_{\overline{T}}^{\overline{G}}(\theta)$ as the alternating sum of the θ-components of ℓ-adic cohomology (compact-supp) of the space $\{g \in \overline{G} : g^{-1}F(g) \in \overline{U}\}$ for full unipotent $\overline{U}$.

Then $R_{\overline{T}}^{\overline{G}}(\theta)$ is a virtual representation of finite G; further:

- each G-irred is in an $R_{\overline{T}}^{\overline{G}}(\theta)$ (for some $\overline{T}, \theta$);
- the various $R_{\overline{T}}^{\overline{G}}(\theta)$ are disjoint (up to "geom conjugacy").

Many G are now well-analyzed; and development continues...

---

[31] E.g. for GL_n, diag-gps via *partitions* of n (here Weyl group $W = S_n$).

(Modular:) Most literature is for repns in *same* char p as G.[32]
Idea: mimic "highest weight" props of Lie-alg repns [Hum78]...
There, roots arise as irred (1-dim) characters of Cartan $\mathcal{H}$;
given simple system $\Pi = \{\alpha_1, \ldots, \alpha_n\}$,
the dual basis $\{\lambda_1, \ldots, \lambda_n\}$ (in usual inner product)
are chars called *fundamental weights*. (list [Hum78, p 69])
For group G? weights should be chars of Cartan subgp H;
with some adjustment for finite fields (e.g. $H = 1$ over $\mathbb{F}_2$).

Indeed for irred V, mimic many standard Lie-alg properties:

- fixed pts V^U (for full unip U) are 1-dim;
- the weight λ on V^U is *highest* in V (in a natural order)
- λ is *dominant*: of form $\sum_{i=1}^{n} a_i \lambda_i$ (here $0 \leq a_i \leq q-1$).

And have bijection: (dominant weights) $\leftrightarrow$ (irred modules) .
Ex: Natl V for $L_3(2)$ has weights λ_1; $\lambda_1 - \alpha_1$; $\lambda_1 - \alpha_1 - \alpha_2$.
Exer 5.2: Explore weights in some more small examples;
e.g. natural for $Sp_4(2)$; 8-dim adjoint for $L_3(2)$; ...

[32] For small "cross"-char repns, cf. [LS74]; +papers of Guralnick-Tiep.

One approach to this high-weight theory:
via restriction from alg-gp—notably by Steinberg [Ste68].
Assume $G = G(q)$ (untwisted) of rank n, with $q = p^m$.
Then $\#(p'\text{-classes}) = q^n = \#(\text{irreds } V_\lambda)$,
for $\lambda = \sum_{i=1}^{n} a_i\lambda_i$ which are *q-restricted*: $0 \leq a_i \leq q-1$.
Indeed, determined by ("basic") irreds for $G(p)$:
Given λ with p-decomp $\sum_{i=1}^{m} p^i\mu_i$ (so μ_i is p-restricted),
Steinberg Tensor Product Theorem [Jan03, 3.17] shows:

$$V_\lambda = V_{\mu_1} \otimes \sigma(V_{\mu_2}) \otimes \cdots \otimes \sigma^{m-1}(V_{\mu_m}) \ ,$$

where σ generates $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ of order m.
Exer 5.3: Give products for the irreds of $L_2(4)$; of $L_3(4)$; ...
Ex: Maxl wt $\sum_{i=1}^{n} (q-1)\lambda_i$ gives the *Steinberg module St*;
- of dimension $|U| = q^m$;
- the unique irred for G which is projective ("defect 0").

(So, = product of all alg-conjs of basic-Steinberg for $G(p)$.)
Exer 5.4: Check *St* of $L_3(2)$ is adjt-8; for $Sp_4(2)$—prod ?

From Lie alg, have *Weyl module* W_λ, with unique irred qt V_λ.
Ex: For $L_4(2)$, adjt $W_{\lambda_1+\lambda_3}$ of dim 15; but $V_{\lambda_1+\lambda_3}$ of dim 14.
Have dimension of W_λ via **Weyl's formula** [Hum78, p 139].
Exer 5.5: (longer) For $Sp_4(2)$, viewed as $O_5(2)$ of type B_2, use formula on W_{λ_1} to get 5-dim orthogonal (reducible!); and on W_{λ_2} to get 4-dim "spin" (symplectic) irred.

Lusztig in 1979 gave a conjecture for the dimensions of the V_λ.
Andersen-Jantzen-Soergel [AJS94] showed
that it must hold for sufficiently large p.
But recent examples of Williamson[33]
show that a lower bound for p must be fairly large.
(E.g. larger than *Coxeter number* h that Lusztig hoped for.)

[33] See `http://people.mpim-bonn.mpg.de/geordie/Torsion.pdf`

A different approach to high-weight theory, using only *finite* G, was developed by Curtis-Richen [Cur70].

The basic context is that of a split BN-pair (lec1,lec4).

Here a *weight* includes not just λ (as a char of *finite* H), but also scalars μ_i roughly recording the effect of the $U_{-\alpha_i} (i \in \Pi)$ on a high-weight vector.

(This compensates for "small" H—e.g. $H = 1$ over $\mathbb{F}_2$.)

Of course, end-results are same as for the alg-gp approach; but sometimes this finite-route can be more convenient.

One result fully generalizing from Lie-algebras [Smi11, 10.1.7]: Given irreducible V_λ and parabolic $P_J = U_J L_j$, the fixed points $V_\lambda^{U_J}$ are irreducible under L_J.

Indeed in the Ronan-Smith presheaf-viewpoint [Smi11, 10.1.8], the irreds/weights $\leftrightarrow$ irreducible presheaves; i.e. choices of irreducible at each *minimal* parabolic.

Exer 5.6: Explore, e.g. $L_3(2)$; $Sp_4(2)$; $L_4(2)$; cf. [RS89].

Now just a few remarks on "small" representations:

For classical matrix groups, the *natural* representation
is the obvious action on vectors.

For exceptional groups, "natural" is often used
to refer to the *smallest* module.
Ex: G_2 on 7-dim Cayley algebra [Hum78, p 105].

Also fairly small is the *adjoint* module—from the Lie algebra.
(For classical, roughly *conjugation* action on matrices.)

The weights on V_λ give orbits under Weyl gp W.
The dominant-weight λ is *minimal* (or *minuscule*)
if just *one* W-orbit in Weyl mod W_λ. (list:[Hum78, p 105])
Then $W_\lambda = V_\lambda$ is irreducible, for any p.
Ex: In type A_m, any λ_i (incl λ_1 for natural).
In type B_m (O_{2m+1}), λ_m gives spin-module of dim 2^m.

The FF context of lec3 gave another notion of "small";
weight-theory is heavily used to get the FF-list [GM02].

§3: Representations for sporadic groups

There is no common *theory* of structure for sporadic groups; much less, for their representations.

But often, get information about some small representations, from their construction(s)—or containments in other groups: (Many such details are in the Atlas(es) [C^+85] [JLPW81].)

- Golay codes: M_{12} in $6/\mathbb{F}_3$; M_{24} in $12/\mathbb{F}_2$.
- Leech lattice: Co_1 in $24/\mathbb{F}_2$; $3Suz$ in $12/\mathbb{F}_3$; $2J_2$ in $6/\mathbb{F}_5$.
- $J_1 < G_2(11)$, so in $7/\mathbb{F}_{11}$.
- $J_2 < G_2(4)$, so in $6/\mathbb{F}_4$.
- $3J_3 < U_9(2)$, so in $9/\mathbb{F}_4$; etc ...

However, some groups have no really small irreducibles; e.g. BM in $4370/\mathbb{F}_2$; M in $196882/\mathbb{F}_2$

We'll PAUSE here; before moving on to applications...

Lecture 5a: The Alperin weight conjecture

(version of 25aug2015)

Stephen D. Smith
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Venice Summer School
7-18 September 2015

Overview of the talk:

Introduction: The Alperin Weight Conjecture
1: Reduction(s) to simple groups
2: A closer look at verification for the Lie-type case
Afterword: A glimpse of some other applications

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

Introduction: The Alperin Weight Conjecture (AWC)

In early 1985, Alperin [Alp87] stated a bold conjecture on the p-modular theory for *any* finite group.

One approach is via simple groups; we sketch some features:

Alperin abstracted the following features of high-weight theory:

- an irred V_λ has a high-weight 1-sp X;
- $N_G(X)$ is a parab $P_J = U_J L_J = N_G(U_J)$, $U_J = O_p(P_J)$;[34]
- $K_J := \langle U_{\pm i} : i \in J \rangle$ is Lie-type, with unique proj-irred St_J;
- and $\lambda \cdot St_J$ is proj-irred for $HK_J = L_J \cong N_G(U_J)/U_J$.

The pair $(U_J\ ,\ \lambda \cdot St_J)$ conversely determines V_λ.

Exer 5.7: Exhibit pairs for $L_3(2)$; $Sp_4(2)$; ... $L_4(2)$?

For any finite H, Alperin defines a *weight* as a pair (P, S); where P is a p-subgp of H (may assume $P = O_p\big(N_H(P)\big)$), and S is a proj-irred module for $N_H(P)/P$. The AWC says:

Conj : For all H,p: $\#$(irreds) $= \#$(weights—up to conj) .

Exer 5.8: Check for A_7 at $p = 2$. (Irreds? Use $A_8 \cong L_4(2)$.)

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[34] The only *p-radicals* $X = O_p\big(N_G(X)\big)$ are U_J—e.g. [Smi11, 4.4.1].

(The AWC has various consequences, e.g. conjs of Broué;
see e.g. Schmid [Sch07, p202].)

For general H, don't get "natural" bijection (as for Lie type).

Some special cases of the Conj (AWC) were verified early on;
e.g. Alperin [Alp87] mentions: solvable, S_n, GL_n, ...

Some approaches proceeded at the level of general H;
esp. topological views—cf. [Smi11, Ch 11].

But for the context of these lectures, we'll focus instead
on approaches via reduction to simple groups.

In particular, we'll explore the motivating case of Lie type G.

Seemingly Alperin did not originally check all Lie types;
soon Cabanes [Cab88] gave a general argument.
(Using the viewpoint of modular *Hecke algebras*.)

We'll re-visit Lie-type later...

§1: Reduction(s) to simple groups[35]

(*Earlier history*) For fuller details, see e.g. [NT11].
The reduction-approach was pioneered by Dade:
who gave variants of the Conjecture (e.g. "projective");
and announced an anticipated proof of the reduction(s).
This led to verifications for many of the simple groups,
by a number of researchers including many of his students.
However, no complete proof of his reduction has appeared.

We mention one feature of verification-proofs for simple G:
For the p-group P to have a projective-irreducible S,
it is necessary that $P = O_p\big(N_G(P)\big)$.
(Recall from lec3: this says P is p-radical.)
So initially one needs to determine the poset $\mathcal{B}_p(G)$ of radicals;
which is also of geom/topol interest—e.g. [Smi11, p 121].

Later came a resurgence of reductions for such conjectures...

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[35] I thank P. H. Tiep for suggestions in this area; more in survey [Tie14].

(*Recent results*) Navarro-Tiep [NT11] give one reduction:
It suffices to show all simple groups G are *AWC-good*;
which is defined by conditions (1.a) $\cdots$ (3.d) in Sec 3 there.
(Indeed (1.b), together with the bijection in 3.2 there,
roughly requires a *partition* of AWC, via p-radical subgroups;
and in fact requires it further for central p'-extensions of G.)
The proof of the main reduction Thm 5.1 relies repeatedly
on consequences of AWC-goodness in Thm 3.2 there.
(And 3.2 in turn parallels Thm 13.1 of [IMN07]—on
goodness for *McKay Conjecture*, which we mention later.)

Since AWC-goodness is stronger that just AWC,
can't just quote existing AWC-verifications for simple G.
E.g. Sec 6 of [NT11] checks goodness for Lie-type in char p.

A related reduction was given by Cabanes [Cab13].
Also Puig [Pui11] has recently reduced a variant of the AWC
to central p'-extensions of simple groups.

§2: A closer look at verification for the Lie-type case

Recall Alperin's motivating situation $(U_J\ ,\ \lambda \cdot St_J)$.
We'll explore a bit, using more of the modular theory above.
(And also make a slight variation at one point...)

We had seen that for untwisted $G/\mathbb{F}_q$ of rank n,
we have $\#(\text{p}-\text{irred}) = q^n$—from Lie-weights, as chars of H.
We will also get q^n, when we use the Alperin-weights:

Recall we had parab P_J stabilizing the high-weight 1-space X.

We will temporarily shift attention
to the "complementary" $P_{\overline{J}}$, for $\overline{J} := \Pi \setminus J$.
This won't change the basic combinatorics of the count...
But it will have the "advantage"
that the $U_{-i} (i \in \overline{J})$ do *not* stabilize X.

We first take the smallest case $q = 2$; to see a special feature:

Here the expected number 2^n is essentially immediate:

For now $H = 1$; so $K_{\overline{J}} = L_{\overline{J}}$, and λ is *trivial* as char of H.

Thus $St_{\overline{J}}$ is the *only* projective-irreducible for $P_{\overline{J}}/U_{\overline{J}} \cong L_{\overline{J}}$.

So one Alperin-weight for each $\overline{J} \subseteq \Pi$—of size n; get 2^n. $\square$

Another consequence of $q = 2$ is that minimal parabolics $P_{\bar{i}}$ have Levi complements $L_{\bar{i}} = K_{\bar{i}} \cong L_2(2) \cong S_3$.

And the only nontriv irred of $L_2(2)$ has dim 2—the (proj!) $St_{\bar{i}}$.

Recall we chose $\overline{J}$ so that $U_{\bar{j}}$ ($j \in \overline{J}$) does not fix X.

So (e.g. using irred-presheaf viewpoint of [Smi11, 10.1.8]),

X under $L_{\bar{j}}$ generates $V_\lambda^{U_{\bar{j}}}$—affording $St_{\bar{j}}$;

and then X under $L_{\overline{J}}$ generates $V_\lambda^{U_{\overline{J}}}$—affording $St_{\overline{J}}$.

Exer 5.9: Explore for $L_3(2)$; $Sp_4(2)$; $L_4(2)$ ...

Thus: using the weight $(U_{\overline{J}}\ ,\ St_{\overline{J}})$ gives us proj-irred $St_{\overline{J}}$: which is now visible, as a subspace—i.e. *within* V_λ.

(Using Alperin's $(U_J\ ,\ St_J)$, we do not "see" St_J within V_λ.)

Now we turn to the general case for q.
In particular, we have $|H| = (q-1)^n$.

For $q > 2$, we can't expect the above "within" property;
so we return to the Alperin-setup of P_J and St_J.
Now we note that $L_J = K_J H = K_J H_{\overline{J}}$,
where $H_{\overline{J}}$ is the product of the H_i $(i \in \overline{J})$ (*not* in K_J).
Weights come from $\lambda \cdot St_J$,
where for λ as a char, $\#(\text{choices}) = |H_{\overline{J}}| = (q-1)^{|\overline{J}|}$.
Combining J-terms over fixed $|\overline{J}| = i$, for $\#(\text{weights})$ we get:

$$\sum_{i=0}^{n} \binom{n}{i} (q-1)^i \, 1^{n-i} = \big((q-1)+1\big)^n = q^n \ ;$$

ideally everybody is at ease with the Binomial Theorem... $\square$

Afterword: A glimpse of some other applications

(1) The **McKay Conjecture** (for ordinary characters, $\sim$1971)
asserts, for finite H with Sylow p-group P, that:

$$\textbf{Conj: } |\mathrm{Irr}_{p'}(H)| = |\mathrm{Irr}_{p'}\big(N_H(P)\big)| \; ;$$

where $\mathrm{Irr}_{p'}$ means the set of characters with p'-degree.

Exer 5.10: Check at $p = 2$ for A_5; A_6; $L_3(2)$ ...

Isaacs-Malle-Navarro in [IMN07, Thm B]
reduce the McKay Conjecture to showing
that simple groups are *good* (for p);
as defined by a list of conditions in Sec 10 of the paper.
Various simple G are treated there and elsewhere...
(We had mentioned that this inspired [NT11] for the AWC.)
Recently Malle-Späth announced the verification for $p = 2$; see
`http://arxiv.org/pdf/1506.07690.pdf`

(2) In the theory of ordinary characters in p-blocks,
the **Brauer Height 0 Conjecture** (1955) asserts,
for a p-block B of H with defect group D, that:

Conj: (irred-degrees in B have p-part $\frac{|G|_p}{|D|}$) $\Leftrightarrow$ D is abelian.

The direction $\Leftarrow$ was reduced to quasisimple groups
by Berger-Knörr in [BK88].
Various cases were then treated—see e.g. summary in [KM13].
Kessar-Malle now complete the treatment in [KM13, 1.1].
(They use the Lusztig-induction approach to characters,
developed after Deligne-Lusztig induction sketched in §2.)

The $\Rightarrow$ would follow—if we knew a strong form (in [NS14])
of the Alperin-McKay Conjecture (blockwise McKay).
This "would follow" in fact comes via [NT13] and [GLP$^+$15];
the latter in turn uses Aschbacher's work
on maximal subgroups of classical groups (later lec6).

(3) Recently Späth announced a similar reduction
for Dade's projective form of AWC...

(4) The **Ore Conjecture** (1951) in group theory asserts that:

Conj: For simple G, every element is a commutator.

(Since the Conjecture is already about simple groups,
there is no question of reduction to simple groups;
instead, CFSG is used only for completeness of the list.)

A substantial literature developed, covering many cases.
Recently Liebeck-O'Brien-Shalev-Tiep in [LOST10]
completed the analysis of the remaining cases.
Their methods are character-theoretic (and use computers);
e.g. via the standard lemma of Frobenius that:

$$g \text{ is a commutator } \Leftrightarrow \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)} \neq 0.$$

Exer 5.11: Explore small cases e.g. A_5, A_6 ...

Some use is made of Deligne-Lusztig theory mentioned above.

THANKS!

Lecture 6: Maximal subgroups of simple groups

(version of 27sept2015)

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Venice Summer School
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Overview of the talk:

Introduction: Maximal subgroups and primitive actions
1: Maximal subgroups of alternating groups
2: Maximal subgroups of Lie-type groups
3: Maximal subgroups of sporadic groups

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

Introduction: Maximal subgroups and primitive actions

Wilson's recent book [Wil09] has a good discussion
of maximal subgroups for various kinds of simple groups.
(He is a major contributor—esp. for maxls in sporadics.)
See also the surveys [KL88] [LS92].
And Aschbacher 2008 Venice summer school lectures:

`http://users.dimi.uniud.it/ mario.mainardis/scuolaestiva2008/venotes.pdf`

To get started, let's first explore how we might approach
finding the maximal subgroups—particularly in a simple G.

I'll discuss some overall features for almost-simple $G = S_n$;
there will be analogues for other G (especially classical).

A structures-list (S):

Our eventual goal is a list of the maximal subgroups M of G.
But more: want to understand them, via some "role" in G...
E.g. often a subgroup will preserve a natural *structure*;
in this case, on the points permuted by $G = S_n$.

Thus an initial goal might be:
(S): List subgroups stabilizing the various structures.
Indeed if we can show any proper subgroup H lies in one such,
then the list gives *candidates* for maximal M.
(This structures-viewpoint is a main theme in today's lecture.)

Some examples of structures—which give standard reductions:
(1) If ≥ 2 H-orbits, then $H \leq$ some $S_k \times S_{n-k}$.
So: reduced to considering *transitive* H. Next—for $1 < k < n$:
(2) If H permutes k-blocks partitioning n, then $H \leq S_k \wr S_{\frac{n}{k}}$.
Exer 6.1: Give examples on 4 and 6 pts (e.g. C_{S_n}(reg inv)).
Thus we've reduced to considering *primitive* H. This suggests:

An actions-list (A): (now for a "general" primitive H)

(For *permutation*-group theory, see e.g. Cameron [Cam99]).
Thus a natural intermediate goal (indep of S_n) might be:
(A): Classify various types of H with primitive action.
These types should give a "qualitative" description of H:
varying notably in terms of the *socle*: soc(H).
(The product of the minimal normal subgroups of H:
it can contain many copies of simple groups.)

Then back to $G = S_n$: try to fit each prim-type for H in (A)
into one of the structure-groups for S_n in (S).
(At which point our maxl-candidate list in (S) is complete.)

I've invented names (S,A), to emphasize the distinction:
since the literature may not clearly distinguish...

§1: Maximal subgroups of alternating groups

(For more detail: e.g. [Cam81] [Pra83][36][Wil09, Sec 2.6].)

As usual, first treat almost-simple S_n; with A_n deduced later.

Much of a structures-list (S) for S_n was known classically:

- mainly "obvious" structures (including (1)(2) above);
- but also: almost-simple groups occur ... "unpredictably".

However, *completeness* of any such list was not known—until:

The O'Nan-Scott Theorem(s!): *One* result with this name is the basic primitive actions-list (A) of Scott [Sco80, p 328].

(As corrected in [AS85, App]; we'll return to (A) soon.)

Scott then [Sco80, p 329] fit the various primitive types in (A) into a structures-list (S) for S_n—which gives further (primitive!) structures (3)–(6), stated on our next page.

The result (S) was also obtained independently by O'Nan; this list for S_n is *also* often called "the" O'Nan-Scott Thm!

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[36] I thank Cheryl Praeger for assistance with this section.

Thm ("S") :(O'Nan-Scott) A proper subgroup H of S_n, other than A_n, lies in one of: (in paren: preserved structure?)

(1) $S_j \times S_k$, where $n = j + k$; (j-set: $\leftrightarrow$ partition j, k)

(2) $S_j \wr S_k$, where $n = jk$; (blocks: 2-dim array $j \times k$)

(3) $S_j \wr S_k$, where $n = j^k$; ("product": k-dim hypercube)

(4) $AGL_d(r) \cong r^d : GL_d(r)$, where $n = r^d$; (affine d-sp/$\mathbb{F}_r$)

(5) $(L^k \wr S_k)\mathrm{Out}(L)$, L simple, $n = |L|^{k-1}$; ("diagonal")

(6) an almost-simple group H. (structure? unpredictable)

Cases (3)–(6) are primitive.

(In (5), $\mathrm{soc}(H)_\alpha$ is a copy of L diagonally-embedded in L^k.)

Exer 6.2: Which of (1)–(6) arise for S_4? S_5? $\cdots$ S_8?

As H in (3)–(6) is primitive, the pt stab H_α is maximal. So: We "reduce" from (maxl for S_n) to (maxl for *all* simple) in (6)!

The "candidates" in (1)–(6) are often but not always maximal; such details were handled, for both S_n and A_n, by Liebeck-Praeger-Saxl [LPS87].

Their result includes partial (but still fairly general) restrictions on the "unpredictable" almost-simple case (6).

More on the actions-list (A):

The actions-list (A) has also been fundamental
for the further study of general primitive groups H.
So for later reference, we'll now summarize certain aspects.

The CFSG is much used for perm-groups (see e.g. [Cam81]);
but it had seemed unneeded for the O'Nan list ... However:

Aschbacher saw Scott had omitted: *twisted* wreath products.
(Not affecting the S_n-list (S)—see TW in discussion below.)
A correction appeared in Aschbacher-Scott [AS85, App];
so (A) is often called the *Aschbacher*-O'Nan-Scott Thm.
(As I will do—to distinguish it from the S_n-structures list.)
Its usual proofs *do* use CFSG—via the Schreier Conjecture.

Below I'll mention several later references on the topic of (A).
Note: these have different subdivisions of primitive-types; so for a comparison, first see e.g. Table 2 in [BPS07] ($\cong$ Table 6.1 in forthcoming Praeger-Schneider book [PS]).

Cameron's view [Cam81, 4.1] has been influential; eg led to a short self-contained proof in Liebeck-Praeger-Saxl [LPS88].
(A *quasiprimitive* (normal subgroups are transitive) version appears in [Pra93]; and does not use Schreier/CFSG.)
A later 8-type subdivision is in e.g. [BP03];
for a more extended, elementary discussion: [PLN97, Sec 6].

The texts [DM96] [Cam99] [PS] have O'Nan-Scott chapters.

For later reference, I'll roughly state (A) via the Table below;
using the 8-type viewpoint of [PLN97, Sec 6].
(I won't try here to give details on those cases;
just their *names*—which are fairly suggestive.)
As in [BPS07], column 1 gives corresp with [LPS88]-cases;
column 3 gives containments in earlier S_n-structures-list (S).

[LPS88]	[PLN97, Sec 6]	$\leq S_n$ str-gp
I	HA (holomorph of el-abelian)[37]	(4)
II	AS (almost-simple)	(6)
III(a)(i)	SD (simple diagonal)	(5)
III(a)(ii)	HS (holomorph of simple)	$<$ (3),(5)
III(b)(i)	PA (product action)	(3)
III(b)(ii)	CD (compound diagonal)	$<$ (3)
$\#(\text{comp}) = 2$	HC (holomorph of compound)	$<$ (3)
III(c)	TW (twisted wreath)	$<$ (3)

[37] The *holomorph* of X is $X \cdot Aut(X)$.

Deduction(s) of the structures-list for S_n

Cols 2,3 above fit the 8 types of (A) into (3)–(6) of (S) for S_n.
(Proving (S) via (A) thus requires Schreier/CFSG;
while using weaker quasiprimitive types [Pra93] would not.)

But *details* for such a deduction from (A) seem hard to find.
E.g. Scott [Sco80, p 329] gives just a 5-line sketch;
and the correction [AS85, App] just mentions TW $\leq$ (3).

More on containments can be found in various early sources;
and later methodically in Praeger [Pra90]. E.g.:

- Standard: TW $\leq$ holomorph of its socle—HS or HC.
- HS,HC $\leq$ PA is detailed in Kovacs [Kov85, Sec 3].
- HS $\leq$ SD, and CD $\leq$ PA, in [Pra90, 3.4,3.9].

Properness ($<$) of containments is usually left implicit;
the reader can use socle str/action for non-isomorphism.

Further relevant discussion appears in [PS].

Wilson [Wil09, 2.6.2] does *not* quote the actions-list (A);
hence also gives a CFSG-free proof of (S) ...

Roughly: for $H(\neq A_n)$ primitive in S_n, he subdivides cases
via partial information on socle-shapes and actions;
rather than fuller information as given in (A).

And in explicit S_n, in effect also via proper-containments,
he reduces his subcases for primitive H down to just (3)–(6).

His main logic-sequence (roughly parallel to [LPS88] for (A)):

First reduce: to non-abelian socle—else (4).

Then, to *unique* minl $\trianglelefteq$-sgp N—for if two, get $H \leq$ a group with their prod as unique minl $\trianglelefteq$. (HC,HS $<$ later gps)

To $\geq$ two components (eg $\cong T$) in N—else (6).

To N_α projecting onto T—else (3). (TW $<$ PA)

A compound-diag is (proper) in (3); (HC $<$ CD $<$ PA)
while a single diagonal gives (5). (HS $<$ SD)

(Thus *implicitly* now, again four primitive-types are non-maxl.)

§2: Maximal subgroups of Lie-type groups

Background: maximal subgroups had been studied earlier
in standard areas such as Lie algebras/groups (e.g. Dynkin),
and algebraic groups (as well as later by Seitz et al...).
See e.g. Liebeck's survey [Lie95].

The finite Lie-type groups were studied via two cases:

- the classical (matrix) groups; and
- "exceptional" groups: usual exceptional $E_6 \cdots G_2$; plus non-classical twisted: ${}^2B_2(Sz)$, ${}^2G_2(Ree)$, ${}^3D_4$, ${}^2F_4$, ${}^2E_6$.

Maximal subgroups of classical groups:

A good discussion of this case appears in [Wil09, Sec 3.10]; we'll largely parallel that viewpoint.

Aschbacher [Asc84] continued for classical matrix groups the viewpoint of structure; now on the natural *module*. (Again basically a reduction to the almost-simple case.)

Maybe easiest to first indicate the case of the linear group; here (Weyl group) S_n-structures suggest $GL_n(q)$-structures. (Recall similar connection in repn theory—lec5.)
We follow Wilson's "elementary" version [Wil09, 3.5]; we write just GL_n for $GL_n(q)$, $q = p^a$. Roughly (!):

Thm:(Aschbacher) Any proper subgroup of GL_n,
not containing SL_n, lies in one of: (preserved structure?)
C_1 (1) $q^{jk}(GL_j \times GL_k)$ (parab!), where $n = j + k$; (j-space J)
C_2 (2) $GL_j \wr S_k$, where $n = jk$; (sum-decomp: $\oplus^k J$)
C_4 (2') $GL_j \cdot GL_k$ (commute), $n = jk$; (tensor-decomp: $J \otimes K$)
C_7 (3) $GL_j \wr S_k$, where $n = j^k$; (k-tensor decomp: $\otimes^k J$)
C_6 (4) $r^{1+2d}Sp_{2d}(r)$, where $n = r^d$;[38] (affine→extraspecial)
$C_?$ (6) an almost-simple group. (*Some* from str; see below)
(Again, "reducing" from classical, to (maxl:all-simple) in (6).)
Exer 6.3: Which arise for $L_3(2)$? $L_4(2)$? ...

Actually Aschbacher defined 8 families C_i of structures;
where first-five above are C_1, C_2, C_4, C_7, C_6. Other 3? in (6):

(C_3) $GL_m(q^r)\mathbb{Z}_r$, where $n = mr$, r prime; (extension field)
(C_5) $GL_n(q^{\frac{1}{r}})$, r prime dividing $a = \nu_p(q)$; (subfield)
(C_8) $Sp_n(q)$, $O_n(q)$, $U_n(q)$. (classical subgp: suitable form)
Exer 6.4: Exhibit C_3, C_8 for $L_4(2)$; C_5 for $GL_2(4)$; ...

But *other* almost-simple? Unpredictable (some say "C_9").

[38] For $r = 2$, replace $Sp_{2d}(r)$ by $O^{\pm}_{2d}(2)$.

Aschbacher treated other classical groups similarly;
defining the C_i suitably with respect to preserved forms.
(E.g. distinguishing isotropic and non-isotropic subspaces...)
See e.g. [Wil09, Ch 3] for statements for Sp_n, U_n, O_n.
Exer 6.5: Explore for S_6—regarded as $Sp_4(2)$; $O_4^-(3)$; ...

The "candidates" in the C_i need not *always* be maximal.
Many details handled by Kleidman-Liebeck [KL90b]. Further:

Maximals for G were *also* described via overlying algebraic $\overline{G}$.
See Liebeck-Seitz [LS98a] for this approach to the C_i.[39]
A crucial situation: $X < Y < \overline{G}$ with X, Y irred on natl $\overline{V}$.[40]
This was treated by Seitz [Sei87] (building on Dynkin).
(With exceptional groups done by Testerman [Tes88].)
For finite G, get results on much of (the char-p subcase of)
the unpredictable almost-simple case "C_9".
(See also the Malle-Testerman [MT11]; Venice school 2017?!)

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[39] I thank Gary Seitz for suggestions in these sections.
[40] Compare with primitive-type inclusions in S_n, in Praeger [Pra90].

Maximal subgroups of exceptional groups:
See e.g. Liebeck-Seitz survey [LS03]; which we mainly follow.

Again focus on structures—now, mainly from the *Lie* theory.
(But also see Wilson [Wil09, Ch 4] and its references; and Aschbacher papers on "natural" module in types G_2, E_6.)
The approach first treated corresponding algebraic groups.

Background: For exceptional Lie (algebras and) groups/$\mathbb{C}$,
Dynkin [Dyn52] gave maximal *connected* subgroups; mainly:

- maxl parabs (building-*sub*substructure—Aschbacher's C_1);
- maxl-rank reductive (root *sub*systems—cf. C_8-simple);
- scattered cases (mostly smallish simple—but eg $F_4 < E_6$).

Exer 6.6: Describe maximal parabolics for G_2; E_6; ...

For the exc algebraic groups, Seitz gave an analogue in [Sei91];
extended to maximal pos-dim by Liebeck-Seitz [LS90].
Originally for char $p > 7$; extended to small p in [LS04].

The algebraic-group results led to results for finite Lie-type:
From field $\mathbb{F}_q$, now expect also cases of "same" Lie type:
 • extension-field, subfield (incl twisted) cases—cf C_3, C_5.
(Like maxl-rank above, part of almost-simple class.) Plus:
 • a few "exotic" local subgroups. (Ex: $2^3 L_3(2)$ in G_2(odd).)
Indeed [LS90, Thm 2] is a structures-list—a maxl sgp must be:
 either from (essentially) one of the above •-structures; or
 • some other, unpredictable almost-simple group. (Cf "C_9")

In further pursuing this last subcase of almost-simple,
 those in char p are *generic* (cf. [LS98b]); else *non-generic*.
Non-generic possibilities are listed in Thm 4 of [LS03];
 and generic cases are indicated in Thms 5,7 there.
Finally Thm 8 of [LS03] summarizes the work above.
 (Also used: Testerman [Tes88]; Liebeck-Seitz [LS99a].)
Some details remain unfinished;
 e.g. conjugacy of some almost-simple cases.

§3: Maximal subgroups of sporadic groups

Many maximal subgroups are described in the Atlas [C^{+}85];
which records the state of knowledge $\sim$1985.
(A summary with fuller references appears in [Wil86].)
Wilson's recent book [Wil09] contains tables
of maximal subgroups for each of the 26 sporadic groups.
These give the status essentially now; in summary:
Those tables are known to be complete—except
possibly for the Monster M.

Sometimes there are visible preserved-structures;
e.g. in Steiner systems/Golay codes, or the Leech lattice.
Ex: stabilizer (Co_2) of "length-2" vector, in Co_1.
Sometimes local subgroups are maximal (often via structure).
Ex: The 2-local $2^4 A_8$ is also the "octad" stabilizer in M_{24}.
But other methods often needed (eg for almost-simple maxls).

We'll PAUSE here; before moving on to applications...

Lecture 6a: Some applications of maximal subgroups
(version of 25sept2015)

Stephen D. Smith
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Venice Summer School
7-18 September 2015

Overview of the talk:

1: Random walks on S_n and generating sets
2: p-exceptional linear groups
3: The probability of 2-generating a simple group

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

§1: Random walks on S_n and generating sets[41]

We summarize some background from Whiston [Whi00]:

Neumann-Praeger [NP92] gave an algorithm for recognizing the span of a set S of elements in a linear group G.
This is based on a *random walk* using S—building up words, via successive multiplication by random choices from S.
Holt-Rees [HR92] adjusted to improve convergence (to full G).
Diaconis and Saloff-Coste [DSC98] specified convergence; using $m(G) :=$ size of maxl independent gen-set for G.
(Independent means: no element is in span of the others.)

Thus it is of interest, particularly for computational efficiency, to determine $m(G)$—esp. for (almost-)simple groups G.

[41] I thank Persi Diaconis for suggesting this topic.

For G given by S_n, Whiston [Whi00, Thm 1] showed:
$m(S_n) = n - 1$ (achieved by usual adjacent-transpositions).
He quotes the (Aschbacher-)O'Nan-Scott actions-list (A).
(This seemingly requires CFSG via Schreier Conjecture.)
But "in effect" he uses the structures-list (S) for S_n.
(Then he quotes *full* CFSG, for case (6) almost-simple.)
Below we give a rough sketch of the argument:

Strategy: From an indep gen-set in S_n, remove an element;
this smaller-span lies in some maximal subgroup $M < S_n$.
(If $M = A_n$? Re-define M, instead removing even-perm.)
To study M, can now use structures-list (S) for S_n.)
So now show: Either $m(M) \leq (n - 3)$;
or $= (n - 2)$—with any further element generating S_n.

The actual logic roughly follows a deduction of (S) from (A):

Reduce to M transitive on n:

else have (1) $S_j \times S_k$—use induction on S_j, S_k; + argt ...

Reduce to M primitive on n:

else (2) $S_j \wr S_k$—similar; induction on S_j, S_k + argt ...

Next quote Aschbacher-O'Nan-Scott—primitive-types for M; but here, only the *group*-structures for M really matter.

(Cf. table in lec6-§1, for action/group correspondence:)

- Types PA,CD,HC,TW: in (3) $S_j \wr S_k$—*group* of (2), done!
- Type HA: in (4) $r^d GL_d(r)$—use lengths of subgp-chains...
- Types SD,HS: in (5) $L^k \wr S_k$—so in $S_{|L|} \wr S_k$; similar ind...
- Type AS: in (6) almost-simple—use minl deg of perm repn; via CFSG + bounds in [KL90b] [C+85]; [CST89] for chains.

$\S$**2: *p*-exceptional linear groups** (Again I thank Tiep.)
We sketch material (already briefly mentioned in lec5)
from Giudici-Liebeck-Praeger-Saxl-Tiep [GLP$^+$15]:

For a linear group $H \leq GL_n(p)$ (irred on natural module V),
one can study special H-orbits on nonzero vectors $v \in V^\#$;
e.g. regular orbits; or at least long orbits; or few orbits; ...

We say H is *p-exceptional* if all orbit lengths are coprime to p;
i.e. each v is fixed by some Sylow-p. (Assume $|H|_p > 1$.)
(Ex: This holds if all orbits have *same* size (as $p \not| \, |V^\#|$);
this is $\frac{1}{2}$-*transitive*—see appl in Thm 6 of [GLP$^+$15].
Also: see appl to $\frac{3}{2}$-transitive, in Cor 7 there.)

There are some obvious cases of p-exceptional; notably:

- H transitive on $V^\#$—$SL(V)$, $Sp(V)$, $G_2(2^a)$ ... (known!)
- $H \leq \Gamma L_1(p^n)$—nonsplit torus, with automorphisms $p|n$.

The main result of [GLP$^+$15] shows
that these are are in fact almost all the primitive examples;
with a partial result covering much of the imprimitive case:

Thm Giudici-Liebeck-Praeger-Saxl-Tiep [GLP$^+$15, Thm 1]
Any $H \le GL_n(p)$ irred, prim, and p-exceptional, is one of:
(i) H transitive on $V^\#$ (known e.g. [Lie87a, App 1]);
(ii) $H \le \Gamma L_1(p^n)$ (determined in [GLP$^+$15, 2.7]);
(iii) $p = 2$, V irred perm S_c, A_c, $c = 2^r - 1, 2^r - 2$; or from:
$SL_2(5) < GL_4(3)$; $L_2(11)$, $M_{11} < L_5(3)$; $M_{23} < L_{11}(2)$.
*Im*prim? [GLP$^+$15, Thm 3] If $H = O^{p'}(H)$, get (i) on comps.
Exer 6.7: Explore the (p'-)orbit sizes in (ii)(iii) ...

Proof? Note $H < GL_n(p)$—else get conclusion (i)-transitive.
Thus H lies in some maximal subgroup M of $GL_n(p)$.
So can quote max-GL_n —in effect: Actually [GLP$^+$15, Sec 12] parallels *deduction*-sequence in Aschbacher [Asc84, Sec 11].
Here is a rough sketch. The first few reductions are easy:
C_1: eliminated by hypothesis of irreducible.
C_2: eliminated by hypothesis of primitive.
(C_8? Cases like $Sp_n(p)$ give concl (i)-transitive as above...)
C_3: Take maxl such fld-ext: so $H \le \Gamma_d(p^{\frac{n}{d}})$ is *absolutely* irred.
Note $d \ge 2$: else in $\Gamma L_1(p^n)$—conclusion (ii).

Now just C_4–C_7, "C_9" remain to be treated.
C_4: By 4.1,4.2, $J \otimes K$ gives only (abs-)reducible cases (above).
C_6: By 6.1, this subfield-case has no p-exceptional examples.
Now focus on soc(H/scalars). If abelian, it has preimage:
C_5: By 7.1, extraspecial gives only imprimitive—excl by hyp.
So non-abelian socle—of form G^k for G simple. If $k > 1$, get:
C_7: By 5.2,5.3,2.8, $\otimes^k J$ gives only cases which are

- either done above—reducible; imprimitive;[42] in (i); in (ii);
- or a case in (iii): $SL_2(5) \trianglelefteq H \cap GL_2(9)$ ($< GL_4(3)$).

Now $k = 1$: so we have H almost-simple;
and "structured" subcases C_8 gave e.g. (i) earlier.
Hence just "C_9"=other almost-simple; apply full CFSG:

- Lie type (same char p): 8.1 shows only get (i)-transitive.
- alternating: 9.1, only (iii): S_c, A_c/perm; or $SL_2(5) = 2A_5$.
- sporadic: 10.1, only (iii): $M_{11} < GL_5(3)$; $M_{23} < GL_{11}(2)$.
- Lie type (char $\neq p$): 11.1 shows only: either (i)-transitive; or (iii): ($\cong$ alt $2A_5$, A_6 above); or $L_2(11) < GL_5(3)$.

[42] Imprim/reducible seems the intended argt for cases in 2.8(ii) there.

§3: The probability of 2-generating a simple group

We summarize some exposition from Liebeck-Shalev [LS95]:

Aschbacher-Guralnick [AG84, Thm B] showed (using CFSG) that any simple group G *can* be 2-generated (cf. lec9a): i.e. $\exists\ x, y \in G$ with $\langle x, y \rangle = G$.

For S_n, Netto had conjectured in 1882 (see [Net64]) that "most" pairs work: i.e. for *randomly* chosen x, y, that probability(generating S_n) $\to 1$, as $n \to \infty$.
Dixon [Dix69] proved this case (before CFSG/O'Nan-Scott!); and then asked the same question for all other simple G.

Various authors, using CFSG, established the general result:
Thm: prob(generating simple G) $\to 1$, as $|G| \to \infty$.

(Already about simple; so not reduction—just list of CFSG.)
In fact, show: prob(*not* generating) $\to 0$. Indeed (cf Sec 1), if generation fails, then $\langle x, y \rangle \leq$ some maximal $M < G$; and we can apply the candidate-lists from lec6.

First notice that we exclude the sporadic groups:
since $|G| \to \infty$ doesn't apply to this finite list of 26.
Summary: The (infinite familes of) simple G were handled:

- alternating: by Dixon [Dix69] (before O'Nan-Scott list).
- classical: by Kantor-Lubotzky [KL90a]:
  Here the argument subdivided via Aschbacher's list,
  i.e. $C_1 \cdots C_8$, and "C_9"=other almost-simple.
  (Cf. similar subdivision that we saw in Sec 2 just above.)
- (remaining) "exceptional": by Liebeck-Shalev [LS95]:
  Here the argument followed the list of (lec6,Sec2-exc);
  so below, we illustrate some of those subdivisions.

Idea: compute prob(gen fails) as the sum, over all M,
of terms: $\text{prob}(x, y \in M) = (\frac{|M|}{|G|})^2 = \frac{1}{|G:M|}^2$.
Or: sum, now over conj.-classes, of just $\frac{1}{|G:M|}$.

A first reduction appears as [LS95, (c), p 110];
cf. earlier [KL90a, (**), p 69]. Roughly:

For G of Lie type over $\mathbb{F}_q$, and a "structured" type M:
the number of classes should be constant (often 1);
and the index $|G : M|$ is a (non-constant) polynomial in q.

So for such M, the sum of terms $\frac{1}{|G:M|}$ should $\to 0$ as $q \to \infty$;
leaving only the sum over (unpredictable) almost-simple M.

In [LS95], this easier sum is over $\mathcal{K}$ ("known"),
including structured-types M from (lec6,Sec 2-exc):

- parab, maxl-rank, scattered, sub-/ext-field, twisted, local;
- plus "large" almost-simple M, of order $\geq$ (about) $|G|^{\frac{5}{13}}$.

(Also used here: Liebeck-Saxl-Testerman [LST96].)

For the remaining sum over $\mathcal{U}$ ("unknown")
with small almost-simple M (under the above $\frac{5}{13}$ bound),
estimates use 2-generation of M including an involution
(from Malle-Saxl-Weigel [MSW94]).

THANKS!

Lecture 7: Geometries for simple groups

(version of 27sept2015)

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Venice Summer School
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Overview of the talk:

Introduction: The influence of Tits buildings
1: The simplex for S_n—as apartment for GL_n
2: The building for a Lie-type group
3: Geometries for sporadic groups

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

Introduction: The influence of Tits buildings

There is a long history of the study of groups
via their actions on suitable *geometries*.
(Notably in algebraic and geometric topology;
for some background see e.g. Adem-Davis [AD02].)

For finite groups, instead of "continuous" geometry,
it is usually natural to study discrete geometries.
E.g. we saw (lec2a) the complex $\mathcal{S}_p(G)$, defined for any group;
sometimes we use this geometry, for the case of simple G.

But today, we focus on set-theoretic *projective* geometry:
For Lie-type groups, the value of this approach
was cogently demonstrated by Tits' theory of buildings.
We had only briefly (lec4) suggested buildings;
today we'll be saying a bit more.

First some historical context:

The classification of semisimple algebraic groups ($\sim$ 1955)
led to unified understanding of the groups of Lie type.
But not yet to a unified *geometric* analysis. Because:

A classical G on V has a natural projective (or polar) space:
the set of all nonzero proper (isotropic) subspaces of V.
But an exceptional G may not have such a "natural" V...

Tits ($\sim$ 1965) found a common structure—including exc'l G:

To motivate: Consider projective space of $G = GL_n(V)$.
The inclusion-chains (recall lec2) give a simplicial complex;
the simplex stabilizers are parabolic subgroups P_J;
and chain-inclusion ($\subseteq$) $\quad\leftrightarrow\quad$ parabolic-inclusion ($\geq$).

The parabolic-complex is common to classical *and* exceptional;
and its properties are axiomatized uniformly.

The building-structure has had far-reaching applications;
see e.g. Abramenko-Brown [AB08, Ch 13,14].

Today we focus on one particular direction of development:
"similar" geometric approaches to *other* simple groups.
For sporadic geometries, we mainly follow the viewpoint
of "Option S" in [Smi11]; cf also [BS08a, Ch 6].

§1: The simplex for S_n—as apartment for GL_n

Again work with almost-simple S_n—on n "points".
One obvious geometry for S_n is:
the $(n-1)$-dimensional simplex Γ, with points as vertices.
Here the geometry of Γ may not really add any new content—
—that is, beyond just the permutation action on the points.
But it's useful as an example, to see some geometric concepts:

Particularly when we regard S_n as the Weyl group of GL_n:
For the boundary of the simplex Γ is a sphere; which gives
an *apartment*—in assembling the building Δ of GL_n.
So by analogy with projective space above: as new-vertices,
take all nonempty proper subsets of the n points; and
consider the complex Σ of inclusion-chains among them.
(The *barycentric subdivision* of (boundary of) simplex Γ.)
We examine some features—via viewpoint+language of Tits:

Example ($n = 3$): The 2-simplex Γ for S_3 on points a, b, c has as boundary a triangle; which subdivides to a hexagon Σ: (For brevity, I omit brace-notation for subsets)

$$\begin{tikzcd}
& a \arrow[r, dash] & a, b \arrow[dr, dash] & \\
a, c \arrow[ur, dash, "a \subset a{,}c"'] \arrow[dr, dash] & & & b \arrow[dl, dash] \\
& c \arrow[r, dash] & b, c &
\end{tikzcd}$$

(For S_4: the subdivision of the surface of a tetrahedron has 24 maximal (2-dimensional) faces. So, hard to draw!)
But the hexagon pictured above (topologically a sphere S^1) should suffice to exemplify most relevant features; e.g.:

Our Σ is (a triangulation of) a sphere, of dimension $(n-2)$.

Indeed, each vertex (subset of points) has a "type":
given by its size. This is one of $\{1, \cdots, n-1\} =: \Pi$.
Hence each simplex σ (subset-chain) also has a type: a $J \subseteq \Pi$.
(And the dimension of that simplex σ is $|J| - 1$.)
The simplices of full type Π are called *chambers*—dim $(n-2)$;
next below, those of dim $(n-3)$ are called *panels*.

Here is a more "group-intrinsic" description of Σ:
Re-define $\Pi := \{w_1, \cdots, w_{n-1}\}$ as usual generating set for S_n:
where w_i is the transposition $(i, i+1)$.
For $J \subseteq \Pi$, define the *parabolic* subgp W_J
as that generated by the w_j for $j \in \overline{J} := \Pi \setminus J$.
E.g. $W_\emptyset = S_n$; $W_\Pi = 1$; and W_1 is a point stab $\cong S_{n-1}$.[43]
Check W_k is a k-set stabilizer; and k-sets $\leftrightarrow$ cosets of W_k.[44]
So get Σ as *coset complex*, on cosets of all the W_J.

[43] For brevity we write W_i for $W_{\{i\}}$.

[44] Posets of stabilizers are examples of more general *subgroup posets* and lattices. (See e.g. many papers of Mainardis!)

"Diagram geometry": Assign the vertex types Π
to the nodes of the Dynkin diagram (of Lie-type A_{n-1}):

$$\overset{1}{\circ} - \overset{2}{\circ} - \overset{3}{\circ} - \cdots - \overset{n-2}{\circ} - \overset{n-1}{\circ}$$

Put your finger on node k—representing a k-subset $\mathbf{k}$;
the "residual" diagram, of type $A_{k-1} \times A_{n-k-1}$,
is the diagram of the stabilizer $S_k \times S_{n-k}$ of the subset $\mathbf{k}$.
Furthermore, it describes the geometric *link* of $\mathbf{k}$
(namely the geometry of all "adjacent" simplices) as the join
of spheres: (Σ_k=subsets of $\mathbf{k}$) $*$ (Σ_{n-k}=supersets of $\mathbf{k}$) !
And similarly for larger simplices...

Exer 7.1: Describe the links of simplices for S_4; S_5; ...

This local diagram-view is a fundamental inductive feature.

The panel stabilizer $W_{\overline{k}} = \langle w_k \rangle$ of order 2 switches
the exactly-two chambers on that panel. (2 gives "thin")

The panel-relation gives paths and distance for chambers:
the *type* of a path is a word in (the generators of) S_n;
with maximal distance $\binom{n}{2} = |\Phi^+|$,
where Φ is the root system of type A_{n-1}.
Exer 7.2: Exhibit-path types in the hexagon-picture for S_3.

Properties similar to those in this section
hold for Weyl groups W of the Lie-types other than A_{n-1};
see e.g. [Car89, Ch 2] [AB08, Ch 1–3].

§2: The building for a Lie-type group

We will first exhibit some Lie-type buildings explicitly;
before commenting on the abstract theory of buildings.
(For an approach focused from the start on diagrams,
see the book of Buekenhout-Cohen [BC13].)

We parallel our view of S_n above: first a "naive" example;
suggesting then the intrinsic-construction via parabolics.
Here the analogue of the sphere for S_n (via subsets)
is the projective space for GL_n (via subspaces).
Indeed let's start with the small example $G = L_3(2)$:

Example: The projective plane for $L_3(2)$

Choose a basis a, b, c for the natural module V of $L_3(2)$.

There are 7 projective points (1-dimensional subspaces);
in abbreviated notation: a, b, c, ab, ac, bc, abc.
And there are 7 projective lines (2-dimensional subspaces);
that is, $\langle a, b \rangle$; $\langle a, c \rangle$; etc.
Further there are 3 points per line, and 3 lines per point;
so 21 edges in our complex Δ. (here: a bipartite graph).

Types Π? Use $\{1, 2\}$—for linear-dim. (Or proj-dim: $\{P, L\}$)

Exer 7.3: Draw the bipartite graph. Observe:

- The "restriction" to a, b, c is the hexagon Σ for S_3.
- Any two edges lie in some hexagon. (apartment!)

Draw similarly for $Sp_4(2)$: now using *isotropic* 2-subspaces;
and symplectic basis with $a \not\perp d$ orthogonal to $b \not\perp c$.
(More exercises can be found in [Smi11, Sec 2.2].)

Next, again much as for S_n:
The correspondence of k-spaces with cosets of stabilizer P_k
suggests the group-intrinsic construction of the building Δ:
(Cf. [Smi11, Sec 2.2] and its references.)

We may as well work in the general case:
Let G be of Lie type; over a field $\mathbb{F}_q$, where $q = p^a$.
(Take G *un*twisted; adjusted remarks will hold for twisted.)
Recall from lec1 the associated structures:
root system Φ, simple system Π, Weyl group W;
unipotent group U, Cartan group H, Borel subgp $B = UH$;
parabolics $P_J = U_J L_J$ for $J \subseteq \Pi$; monomial $N = N_G(H)$.

We obtain the *building* Δ as the complex given by
the cosets of the parabolic subgroups P_J ($J \subseteq \Pi$).
(As $N_G(P_J) = P_J$, can use conjugates rather than cosets.)
Exer 7.4: Exhibit for $L_3(2)$; observe $\cong$ with proj plane.
We'll now indicate some of its fundamental properties:

Some of these we previewed already for S_n:

We get Δ of dimension $|\Pi| - 1$;
now it is composed of *many* spheres (cont'd next page).

Again each simplex has a type $J \subseteq \Pi$; incl chambers, panels.

And the diagram-inductive property! E.g. for $A_{n-1} = GL_n(V)$:

$$\overset{1}{\circ} - \overset{2}{\circ} - \overset{3}{\circ} - \cdots - \overset{n-2}{\circ} - \overset{n-1}{\circ}$$

This time, the subdiagram of type $A_{k-1} \times A_{n-k-1}$ is
the diagram of the stabilizer of a k-space V_k.
(A parabolic P—with $P/O_p(P) \cong GL_k \times GL_{n-k}$.)
It *also* describes the geometry of the link of V_k, as a join:
(Δ_k=subspaces of V_k) $*$ (Δ_{n-k}=superspaces of V_k).

Now, the panel stabilizer $P_{\overline{k}} \geq L_{\overline{k}} \cong SL_2(q)$—transitive on
the $q+1$ chambers on the panel. ("thick": as $q + 1 > 2$)[45]

Again we get paths and distance for chambers:
with path-types still given by words in W;
and maximal distance $|\Phi^+|$.

[45]Put $q = 1$ in G, Δ-expressions to get analogues for W and "thin" Σ.

There are further geometric relations between W and G:

A chamber stabilizer is given by $P_\Pi = B$.
The N-orbit of B is a single sphere:
namely the Coxeter complex Σ for W;
called an *apartment* of the building Δ. Important property:

Any two chambers are contained in some common apartment.

The chamber "B" is on $q^{|\Phi^+|}$ apartments; each is determined by its unique chamber at maximal distance $|\Phi^+|$ from B.
(Topologically, Δ is a "bouquet" of such spheres.)

These facts in turn reflect some group-theoretic relations:
We have a double-coset decomposition $G = BWB$.
(Shorthand for BNB, where $N/H \cong W$ with $H \leq B$.)
(And elements get canonical form, *Bruhat decomposition*.)
Indeed for parabolics we similarly have $P_J = BW_JB$.

The material in the section so far
has focused on properties of buildings in Lie-type groups.
This is the kind of information usually required
for problems involving applications of simple groups.

Thus we'll only briefly mention Tits' general theory:
which defines buildings as abstract chamber complexes,
via axioms based on the complex Σ for a Coxeter group W.
See e.g. [AB08, Sec 4.1] (and compare [Ron89, Sec 3.1]).

The general theory has been applied in various significant ways;
again see [AB08] for some directions.
For our finite-simple-group purposes here, the crucial result
is Tits' classification [AB08, Ch 9] [Ron89, Ch 8]; roughly:

Thm: A finite building (i.e. "spherical" Σ) of rank at least 3
comes from a Lie-type group over a finite field $\mathbb{F}_q$.

Thus our more explicit material earlier in the section
in fact includes the finite case of abstract-buildings.

§3: Geometries for sporadic groups

The success of Tits' theory of buildings ($\sim$ 1965)
led to much further geometric analysis in group theory.
The later viewpoint in Tits' "local approach" paper [Tit81]
was very influential; esp. the diagram-inductive property.

One direction was to search for geometries for *sporadic* groups;
this analysis was pioneered by Buekenhout, notably [Bue79].
Because of Tits' classification of finite buildings above,
the search for new geometries other than buildings suggested
using rank-2 geometries other than generalized polygons.
(Bldg of GL_3 is gen'd triangle; for Sp_4, gen'd quadrangle; ...)

Buekenhout used e.g. the *circle* geometry on a set S:
vertices are given by: elements, and element-pairs, from S.
He was able to give "diagram geometries"
for many of the sporadic groups.
This work inspired a great deal of further research...

The diagram-inductive feature of these sporadic geometries gives partial analogues of type(etc)-properties of parabolics.
However, again because of Tits' classification of buildings, we cannot expect real analogues of apartment structures.

But the parabolic-analogy above can be further developed: seek geometries in which stabilizers are p-local subgroups.
This is not usually the case in Buekenhout's geometries; but holds in the 2-*local geometries* of Ronan-Smith [RS80].
For details see e.g. [Smi11, Sec 2.3] (also [BS08a, Ch 6]).
Today we'll just extract a few high points:

The first 2-local geometry found was for M_{24}:

Some background (e.g. [Con71, p 225]):

Here $G = M_{24}$ preserves a Steiner system $\mathcal{S}(5,8,24)$:
 a collection of 759 special 8-sets (called *octads*);
 where any 5-subset of 24 lies in exactly one octad.

Also have 8^3-partitions of 24 via octads, called *trios*;
 and 4^6-partitions (any pair an octad), called *sextets*.

The stabilizers P_O, P_T, P_S of an octad, trio, sextet turn out to be 2-local subgroups;[46] with structure:

$$2^4 : L_4(2), \quad 2^6 : \big(\ L_2(2) \times L_3(2)\ \big), \quad 2^6 : 3Sp_4(2)\ .$$

These "look like" parabolics P_J, in Levi decomposition $U_J L_J$.

Ronan-Smith: The diagrams can be combined [Smi11, p97] as sub-diagrams of a larger "Dynkin-like" diagram:

$$\overset{O}{\circ}\!=\!\overset{T}{\circ} - \overset{S}{\circ} - \square\ ; \qquad\qquad \text{(compare type } C_4)$$

where the $\square$ indicates we should *not* expect a group "$P_\square$".

[46]They fix spaces of dim 1,2,4—in 11-dim Golay-code module over $\mathbb{F}_2$.

Correspondingly, the *2-local geometry* for M_{24} is the complex
on vertices given by octads, trios, and sextets;
with the obvious incidence among vertices of different types.
The geometry has the diagram-inductive feature:
For example, the subdiagram $\circ - \circ - \Box$ for an octad O
expresses the geometry of trios and sextets on O:
15 proj points T, and 35 proj lines S; but *not* planes.
(This is a rank-2 *truncation* of proj 3-space for $L_4(2)$.)
Similarly, the subdiagram $\circ \quad \circ - \Box$ for a trio T
describes the link of T as the join:
(3 octads in T) $*$ (7 sextet-partitions refining T) .

This approach led to 2-local geometries for other sporadics.
(Indeed formally, for all—in [BS08a, Ch 6];
but some of those have little real geometric content.)

A one-node extension leads from M_{24} to Co_1; and then to M.
A similar series proceeds from M_{22} to Co_2 to BM;
these use a rank-2 *Petersen geometry*, with vertices given by the 10 nodes and 15 edges of the classical Petersen graph.
(See also Ivanov-Shpectorov papers, e.g. [IS94].)

Some other unusual geometries were discovered in these years:
mostly for individual simple groups of various types,
and in a fairly "sporadic" fasion.
We'll be mentioning a few of those, as we continue.

Lecture 7a: applications of geometric methods

(version of 25sept2015)

Stephen D. Smith
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Venice Summer School
7-18 September 2015

Overview of the talk:

1: Geometry in classification problems
2: Geometry in representation theory
3: Geometric decompositions of group cohomology

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

§1: Geometry in classification problems

The discovery of new geometries suggested the possibility
of classifying collections of geometries
properly including the finite buildings.
This turns out to be hard! (But there have been some results.)

One extension class is now usually called *Tits geometries*;
originally "type M" in Tits' local-approach paper [Tit81].
Roughly, these require (cf. [Smi11, 2.2.34])
that rank-2 "residues" should be generalized polygons;
giving a Cartan matrix M, with a Coxeter-diagram;
but *not* requiring properties of apartments as in buildings.

Tits-geoms exclude most examples in lec7-Sec 3 for sporadics; but include other unusual cases: (using a $\sim$-notation below)

- Neumaier's C_3-geometry for A_7, e.g. [Smi11, 2.3.7]. ($\overset{P}{\circ} - \overset{L}{\circ} = \overset{\pi}{\circ}$: On 7 points; all $\binom{7}{3} = 35$ 3-sets as lines; and 15 (of 30) projective planes on these points,lines.)
- $\tilde{C}_2$-geom for Suz; $\tilde{G}_2$-geom for Ly (get "sporadic bldgs").
- $\tilde{X}$-cases for some classical groups [Smi11, 9.3.9].

The $\sim$-cases were unexpected for *finite* groups—
—since they correspond to infinite *affine* Weyl groups $\tilde{W}$: (For further discussion see e.g. [Smi11, pp 290ff].)

A finite Weyl group W has an infinite affine extension $\tilde{W}$: see e.g. [GLS98, p 12] for the list, with extended diagrams.

Bruhat-Tits: A building of type W over a local field (e.g. $\mathbb{Q}_p$) gives a building of type $\tilde{W}$ over the residue field (e.g. $\mathbb{F}_p$).

Should the finite $\tilde{X}$-examples arises as quotients of the latter?

Some classifications by Tits: First e.g. [Ron89, 7.9]—roughly:
Thm: The universal cover of a Tits-geom (rk ≥ 3) is a bldg.
(Harder if A_3, C_3, H_3(dodecah) involved. Cf. C_3/A_7 above.)
Next e.g. [Ron89, 10.25]—roughly:
Thm: An affine bldg (rk ≥ 4) arises as above (local-field).

This applies to many finite "affine" cases on previous page;
and Kantor [Smi11, 9.3.10] specified the explicit covers.
There is further literature in this direction; e.g. [KLT87].

We briefly mention some other classification-type results:

Timmesfeld [Tim83] et al used group-theoretic hypotheses,
to produce some classifications of finite groups with
subgroups defining a Tits-geometry and diagram.
During the process of proofs, cases for the small field $\mathbb{F}_2$
led to finding some as-yet-unknown "sporadic" geometries!
(later explained via Kantor's coverings—see above)
Onofrei [Ono11] extended the approach to fusion systems.

§2: Geometry in representation theory

We should perhaps first recall from lec4 that
algebraic geometry is basic for Deligne-Lusztig theory
of ordinary representations of Lie-type groups.
But we won't pursue that direction here,
since the focus of this lecture is on projective geometry.

Instead we examine some application-areas for simple groups:
first viewing the "ideal" geometry of the Lie-type building;
then turning to possible analogues for sporadic geometries.

First recall some complexes from collections of p-subgroups:
From lec 5, the p-radicals $\mathcal{B}_p(G)$ (where $X = O_p\big(N_G(X)\big)$).
For Lie-type G, get the unipotent radicals [Smi11, 4.4.1];
so $\mathcal{B}_p(G)$ gives another view of the building Δ.
Further $\mathcal{B}_p(G)$ is homotopy-equivalent [Smi11, 4.3.4]
to $\mathcal{S}_p(G)$ (p-subgroups) and $\mathcal{A}_p(G)$ (elem ab) of lec2a.
Some sporadic geometries are equivalent to these complexes.

The generalized Steinberg module and projectivity

We had mentioned in lec2a the Brown-Quillen result that:

$\tilde{L}\big(\mathcal{S}_p(G)\big)$ is a (virtual) projective module.

This is often called the *generalized Steinberg module* for G:

For G of Lie type, using equivalent $\mathcal{B}_p(G)$ (i.e. building Δ), it does in fact give the usual Lie-type Steinberg module; via the "Solomon-Tits argument" [Smi11, 3.4.15]. Sketch:

A chamber "B" is on $q^{|\Phi^+|}$ apartments. These will give basis: Each sphere Σ adds 1 dimension in top homology $\tilde{H}_{|\Pi|-1}$.

To show homology vanishes in lower homology in the alt-sum: form Δ^- by removing the $q^{|\Phi^+|}$ maxl-distance chambers; and then show Δ^- is contractible.

For contractibility, use the "gate" property of buildings:
Each panel has *unique* chamber closest to B.
So in Δ^-, farthest panels π lie on just one chamber c;
hence can "collapse" c: from π to rest of bdry(c).
Repeating, for each smaller distance, gives contraction.

Exer 7.5: Use the $L_3(2)$-graph of Exer 7.3,
say with initial chamber $a \subset \langle a, b \rangle$,
to verify contractibility of Δ^-, as in Solomon-Tits argt.
The 8 maxl-dist chambers give $\mathbb{F}_2$-basis for Steinberg-module.
(Similarly explore $Sp_4(2)$-graph and Steinberg of dim 16.)

For other G: the projective $\tilde{L}(\mathcal{S}_p(G))$ is usually not irreducible.

Using other complexes Δ: Webb extracted sufficient conditions for projectivity of $\tilde{L}(\Delta)$; notably [Smi11, 4.3.4]:

(*) OK, if contractible fixed points Δ^P for p-groups P.

(This may lead to homotopy-$\simeq$ with $\mathcal{S}_p(G)$ [Smi11, 4.4.12].)

Projectivity for various sporadic G and p-local Δ was verified via (*) in Ryba-Smith-Yoshiara [RSY90]; e.g. 2-local M_{24} ($\simeq \mathcal{S}_2(G)$); C_3 for A_7 (proj; $\not\simeq \mathcal{S}_2(G)$).

Some Δ have $\tilde{L}$ with weaker property: *relative* projectivity; see e.g. Maginnis-Onofrei [MO09].

Chapter 6 of [Smi11] surveys further literature on projectivity.

Finally: for general finite G, the generalized Steinberg module, and the underlying chain complex developed by Webb, are used in topological contexts; see e.g. Grodal [Gro02].

Irreducible modules and local coefficients

Again start with the case of G of Lie type; on V in natl char p.
In lec5, we had briefly mentioned [Smi11, 10.1.7]:
For V irreducible under G, also V^{U_J} is irred under L_J.

When applied for a single P_J, the result has various uses; mainly mod-repn theory of Lie-type and algebraic groups—eg:

- For GL_n, S_n: cf. Kleshchev [Kle97] on branching rules (decomposing restrictions of irreducibles to subgroups).

But we also mention some "adjacent" areas of application:

- For maximal subgroups: e.g. Liebeck-Saxl-Seitz [LSS87]; in setup (lec6) of irred $X < Y < G$ classical on natl V.
- For p-compact groups: (analogue of compact Lie gps) e.g. Andersen-Grodal-Møller-Viruel [AGMV08]; applied to Steinberg V, restricted to elem p-groups.

When the result is applied for all J: the maps $P_J \mapsto V^{U_J}$ define a *coefficient system* (or *presheaf*) on the building Δ.
This viewpoint also has applications in various directions:

(For background and more development, see [Smi11, Ch 10].)
Ronan-Smith [RS85] observed there is a 1:1 correspondence:
(irreducible modules) $\leftrightarrow$ (irreducible presheaves) ;
and approached the irreducibles via homology of presheaves.

One application is to *embedding* a geometry Δ in V;
i.e. "points, lines" of Δ mapped to proj points, lines of V.

- For Lie type: e.g. gen'd hexagons, Cooperstein [Coo01].
- For sporadic: e.g. 2-local M_{24} in Golay-code V, lec7-Sec3.

Exer 7.6: Show C_3 for A_7 is not embeddable—in any $V/\mathbb{F}_2$:
(Recall lines are *all* 3-sets of 7 points; check that
setting (3-set sums $= 0$) forces point-vectors to also be 0.)
For various results on embeddings, see e.g. [Smi11, pp320ff].

We mention that the coefficient-system viewpoint is also used
in Grodal's approach [Gro02] to higher limits in topology.

§3: Geometric decompositions of group cohomology

(For more on this topic, see e.g. [Smi11, Ch 3] [BS08a, Ch 5].)

Webb observed [Smi11, 7.2.5] for (suitable) projective $\tilde{L}(\Delta)$, applying Ext^* gives a decomposition of group cohomology:

$$H^*(G)_p \;=\; \bigoplus_{\sigma\in\Delta/G} (-1)^{\dim\sigma}\; H^*(G_\sigma)_p \;.$$

Sometimes the formula is used for explicitly computing H^*; eg Adem-Maginnis-Milgram [AMM91] used 2-local geom M_{12}.

But its major influence was to motivate ($\sim$ 1990s) "underlying" decompositions of the classifying space $BG_p^\wedge$.
These are *homotopy colimits*, over standard posets like $\mathcal{S}_p(G)$; hence they apply to any finite group G.
(Such work of Jackowski, McClure, Oliver, Dwyer and others is summarized (in a group-theory view) in [BS08a, Ch 5].)
For sporadic G, Benson-Smith [BS08a, Ch 7] show 2-local Δ is homotopy-equivalent to one of the standard posets; hence get such a decomposition of cohomology for G.

We also mention some geometric approaches
to the Alperin Weight Conjecture:
(For more on this topic, see e.g. [Smi11, Ch 13].)
We saw (lec2a) that AWC involves the p-radical poset $\mathcal{B}_p(G)$.
Influential work of Knörr-Robinson [KR89]
uses an equivalent complex Δ (see e.g. [Smi11, 4.6.2]):
of chains of p-subgroups each normal in the last.
They consider AWC partitioned over the various p-blocks B:
For defect-0 B, the unique (proj) irred gives the requirement.
For $\mathrm{def}(B) > 0$, they require vanishing of the alternating sum:

$$\sum_{c \in \Delta/G} (-1)^{\dim c} \, |\mathrm{Irr}(B_c)| \ ;$$

where B_c lifts to B in the standard Brauer correspondence.

This formula is in fact the degree-term of fixed-points $\tilde{L}(B)^G$,
in their *Lefschetz conjugation module* $\tilde{L}(B)$:
the alternating sum of the $\mathrm{Ind}_{G_c}^{G}(B_c)$—under *conjugation*.
Paralleling Webb, they show in fact that $\tilde{L}(B)$ is projective.
This suggests an analogy for *module* cohomology of B:
that $H^*(G, B)$ should decompose via the $H^*(G_c, B_c)$.
Indeed this holds for $H^{>0}$; but maybe as hard as AWC for H^0.
For recent developments related to this approach, see the talk:
`www.math.uic.edu/~smiths/talkl.pdf`

The various applications of $\mathcal{B}_p(G)$ etc have motivated
their determination for many simple G, especially sporadics.
We mentioned (lec5) work of students of Dade;
see also papers of An, O'Brien, Yoshiara, Sawabe, et al.

Lecture 8: Some fusion techniques for classification problems

(version of 27sept2015)

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Venice Summer School
7-18 September 2015

Overview of the talk:
1: Glauberman's Z^*-Theorem
2: The Thompson Transfer Theorem
3: The Bender-Suzuki Strongly Embedded Theorem

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

Introduction

Rather than examining further properties of simple groups,
today we consider some influential early results on 2-fusion.
These results quickly came to be regarded as fundamental,
and were very frequently applied, throughout the CFSG.
We'll overview some of those applications,
and suggest similar uses in other types of problems.

Thereafter we consider the possibility of extending each result
to a suitable analogue for odd primes p.

§1: Glauberman's Z^*-theorem

Often this begins the work on a classification problem:
by showing there must be "some" fusion;
namely conjugacy among involutions in a Sylow T.

Glauberman's general result can be stated [ALSS11, B.2.1]:
Thm: If invol z of H commutes with no distinct conjugate,
then $z \in Z^*(H)$—the preimage of $Z\big(H/O_{2'}(H)\big)$.
Exer 8.1: Check for H dihedral, with $|H|_2 = 2$ or 4.

When H is actually simple G, then normal $Z^*(G)$ is trivial.
So get commuting $z \neq z^g$. Then $\langle z, z^g \rangle$ is in some Sylow T.
Thus have G-fusion of z inside T—indeed even in $C_T(z)$.

Proofs? Glauberman [Gla66] used 2-modular repn theory.
Waldecker's recent [Wal13] emphasizes local group theory;
but (roughly) assumes known involution centralizers.

See also: Broué [BT83] Rowley [Row81] Toborg [Tob] .

Below, we indicate some (historically significant) applications.

For a problem which reduces to a simple counterexample G, as above we immediately get $z \neq z^g \in C_T(z)$.

How does this help? Depends on hypotheses... My favorite:

Application to large extraspecial subgroups

Recall (lec2) Even Case Ex; and summary of "$GF(2)$-Type".
Below are a few features; cf. [ALSS11, Ch 7; esp 7.0.5]:

Here G simple, z invol, $M := C_G(z)$, $Q := O_2(M)$. Assume:
Q is extraspecial ($Q' = \Phi(Q) = Z(Q)$ is of order 2);
and "large": $F^*(M) = O_2(M) = Q$.

By Z^*-thm, have $z, z^g =: a \in M$. How use this? Roughly:

May take $a \in Q$ (else done, by Aschbacher [ALSS11, 7.2.3]).
In el ab $\tilde{Q} := Q/\langle z \rangle$, study action of $\overline{M} := M/Q$ on $\tilde{a}$.
E.g. for $Q_a := Q^g$, $A := Q \cap Q_a$ is elem ab ($A' \leq \langle z \rangle \cap \langle a \rangle$);
and $L := Q(Q_a \cap M)$ has $\overline{L}$ elem ab, normal in $C_{\overline{M}}(\tilde{a})$.
Exer 8.2: Exhibit these, in the case $G = L_5(2)$.

Timmesfeld [ALSS11, 7.3.1] determined cases for $\overline{M}$;
Smith and others [ALSS11, 7.0.1] found corresponding G.

Application to the Sylow 2-subgroup of $U_3(4)$

As part of "small" subcase $m_2(G) \leq 2$ of the Odd Case (lec2), Lyons [Lyo72] characterized $U_3(4)$ by its Sylow 2-group T. (This is in effect a recognition theorem, as in lec4.)

Sketch: The only invols of T are z_1, z_2, z_3 (giving $Z(T)^{\#}$).
Lyons [Lyo72, Lm 1] uses Z^*-thm in simple G with Sylow T, to get z_1 conjugate to another involution of T.
The only other choices are z_2, z_3; so z_1 conj to (say) z_2.
But the Z^*-thm similarly shows z_3 is conjugate to z_1 or z_2.
So all 3 are in fact conjugate.

This then leads to further information...
E.g.: By Burnside's Fusion Theorem [GLS96, 16.2], this conjugacy in $Z(T)$ is induced by $N_G\big(Z(T)\big)$.
So $N_G\big(Z(T)\big)$ has a 3-element (of S_3 on the z_i).
Again, this is a start; of course much more work remains...

Application to semi-dihedral and wreathed Sylows

Classification of simple G with the above Sylow 2-groups T
was another part of the small Odd subcase $m_2(G) \leq 2$;
done by Alperin-Brauer-Gorenstein [ABG73].

Their intermediate analysis treats certain non-simple $H \geq T$,
called "Q-groups" (I won't try to define them here);
to which they apply the Z^*-thm in the "forward" direction:

For such T, $Z(T)$ is cyclic. By (ii) in their Props 1.1,1.2,
subgroups of $Z(T)$ are weakly closed in T (wrt such H).

At 3.1, they get $H = O_{2'}(H) C_H\big(Z(T)\big)$. How?

By ind, $O_{2'}(H) = 1$. So need $Z(T) \leq Z(H)$. Thus WMA:
$Z_0 := Z(T) \cap Z(H) < Z(T)$. Take Z_0 index-2 in $Z \leq Z(T)$.
So in $\overline{H} := H/Z_0$, $\overline{Z}$ is generated by an involution.
Subgroup weak-closure (ii) above leads to $\overline{Z}$ w cl in $\overline{T}$.
So the (forward) Z^*-theorem forces $\overline{Z} \leq Z(\overline{H})$.
Standard argts give $Z \leq Z(H)$—contrary to defn Z_0. $\Box$

§2: The Thompson Transfer Theorem

Typically this is used to eliminate non-perfect "shadows";
namely H which are locally similar to simple examples G.
But in effect (cf. the Z^*-theorem), the contrapositive
again forces some involution-fusion in simple G;
this time, into subgroups of index 2 in T.

An elementary statement (cf. [ALSS11, B.2.9][GLS96, 15.16]):

Thm: If inv z has no H-conjugate in T_0 of index 2 in Sylow T,
then $z \notin O^2(H)$. (So $H > O^2(H)$, and H not simple.)

Exer 8.3: Check, for a transposition z in non-simple $H = S_n$.

(The proof is elementary—directly compute the homological[47]
"transfer" homomorphism: in effect, of H/H' into T/T'.
See also "control of transfer", e.g. [ALSS11, p271].)

Again we give some representative applications, inside CFSG.
A number are indicated in the exposition of [ALSS11];
e.g. look up "Thompson Transfer" in the index there.

---

[47] Cf. notes at `www.math.uic.edu/~smiths/transfer.notes.pdf`

Some applications to quasithin groups

More than 30 to choose from! Here is a fairly typical one:

Notice quasithin $U_4(3)$ has a 2-local $L \cong 2^4 : A_6$.
How elim $2^4 : S_6$, in "shadow" $U_4(3)\langle t \rangle$? At [AS04d, 13.5.16]:
Get much local structure in G as it *would* be in $U_4(3)\langle t \rangle$:
For $z \in 2^4$, set $H := C_G(z)$ and $Q := O_2(H)$ ($\cong 2^{1+4}\langle t \rangle$).
Get $Z(Q) = \langle z, t \rangle$, and $N_G(Q) = H$;
and get all invols x in $T \cap L$ (index 2 in T) conj to z.
Exer 8.4: Check these facts, in actual $U_4(3)\langle t \rangle$.
Also, get Q is weakly closed in H (hence T) wrt G.
So by the Burnside Fusion Thm mentioned earlier,
G-fusion in $Z(Q)$ is induced by $N_G(Q) = H$.
Now $z \in Z(H)$ can't be H-conjugate to $t \neq z$.
So t can't be G-conjugate to any inv $x \in T \cap L$ by above.
Then by Thompson Transfer, $t \notin O^2(G)$—so G not simple. $\square$
(Case t not-invol is also handled there via Thompson transfer.)

Important applications arise in proving [AS04d, Thm 2.1.1]:
the QT-analogue of the Global $C(G, T)$-Theorem (lec3), dealing with T in a unique maximal 2-local M of G.

E.g.: How eliminate shadow of $L_3(2^n)\langle x\rangle$, for graph-aut x?
From local with $L_2(2^n)$ on natl N, set $R := NN^x$;
so $R\langle x\rangle$ (essentially) gives T, and $N_G(T)$ its unique M.
Much work leads to the fusion result 2.4.21.2 there:
For i any involution of R, we get $i^G \cap T \subseteq R$.
Exer 8.5: Check this holds in actual $L_3(2^n)\langle x\rangle$.
Thus x is not G-fused to any $i \in R$.

Here R is of index 2 in T, so using Thompson Transfer we get $x \notin O^2(G)$ at [AS04d, 2.4.22.2].

If non-graph? x involving field-automorphisms is eliminated by a similar Thompson-Transfer argt after [AS04d, 2.4.24].
(This corrects "in" 2.4.24 at discussion [ALSS11, p 99].)

An application to 4-groups[48] and graph connectivity

Our sketch (lec2) of trichotomy omitted various details;
notably for connectivity of rank-3 graph via rank-2 groups.

E.g.: The technical result [ALSS11, B.4.9] is applied,
re connectivity, after 1.5.1 and for B.4.10 there.

That result assumes $G = O^2(G)$ and $m_2(G) \geq 3$; and shows
that if a 4-group V has $m_2\big(C_T(V)\big) = 2$ ("isolated" in T),
then some conjugate $V^g \leq T$ has $m_2\big(C_T(V^g)\big) \geq 3$.

Roughly: Take a 4-gp A normal in T; using $m_2(G) \geq 3$,
get $T_0 := C_T(A)$ of index 2 in T, with $m_2(T_0) \geq 3$.

Show such a V is $\langle z, v \rangle$, with z *unique* inv in $R := C_{T_0}(v)$.

Get $V < C_T(V) = \langle v \rangle \times R$; so z is a square in R.

As $G = O^2(G)$, Thompson Transfer gives
some $v^g \in T_0$—indeed "extremal": $C_T(v)^g \leq T$.

Then z^g is a square in $R^g \leq T$; so $z^g \in T_0$, by index 2 in T.
That is, $\langle z^g, v^g \rangle = V^g \leq T_0 = C_T(A)$.

Thus $m_2\big(C_T(V^g)\big) \geq 3$—via $V^g A$, or $C_T(A)$ if $V^g = A$. $\square$

[48] rank-2, i.e. $\mathbb{Z}_2 \times \mathbb{Z}_2$; also often called *fours-groups*.

§3: The Bender-Suzuki Strongly Embedded Theorem

We already mentioned this fundamental result in lec2.
In today's view, it exhibits "strong control of fusion",
by a single subgroup M (containing $N_G(T)$).

We recall first that the simple groups G in the conclusion
are the Lie-rank 1 groups $L_2(2^n)$, $U_3(2^n)$, $Sz(2^{\mathrm{odd}})$.
And we considered the hypothesis in the form:

- The commuting graph on involutions is disconnected.

(We also mentioned the version, strongly embedded $M \geq T$:

- For $g \in G \setminus M$, $M \cap M^g$ has odd order.)

Other equivalent hypotheses appear e.g. in [ALSS11, B.4.7].
Exer 8.6: Check some: e.g. exhibit the M, in $L_2(2^n)$ ... above.

As before, we explore some representative applications.

Application(s) related to connectivity

Uses with signalizer functors (cf. lec2) are arch-typical.
But other kinds of connectivity arise, roughly based on:
component-stabilizers lying in a common $M < G$.
Relationships among various relevant graphs
are discussed e.g. beginning at [ALSS11, p 36].

For example: in Aschbacher's result [ALSS11, 1.5.10],
the role of the M containing component stabilizers
is played by the *2-generated core* $\Gamma_{2,T}(G)$:
generated by $N_G(S)$ for $T \geq S$ of rank ≥ 2.
When $m_2(G) \geq 3$, beyond the Bender groups only J_1 arises.

How? Aschbacher at [Asc74, 3.7] reduces to:
$C_G(z) \leq M$ for *2-central* z—for otherwise,
we get "new" conclusion J_1 arising here, by his earlier 2.5[49]
(which uses a previous classif-result by Gorenstein-Lyons).
If x is another involution of T, and we have $C_G(x) \not\leq M$,
then by his 3.8, we get $C_M(x)$ strongly embedded in M.
This gives the sufficient local-condition in his [Asc74, Thm 2],
to force M strongly embedded in G (incl $C_G(x) \leq M$).
So the "old" rank-1 Lie-type conclusion groups now arise. □

[49] Note that "3.5,2.1" just before [Asc74, 3.7] should be "3.3,2.5".

Application to quasithin groups

Aschbacher-Smith analyze QT under "even characteristic":
in particular, 2-central z has $F^*\big(C_G(z)\big) = O_2\big(C_G(z)\big)$.
But [AS04d, Ch 16] then extends to "even type" of [GLS94],
where $C_G(z)$ might have certain components L.
The only new group to arise is J_1. (Cf. [ALSS11, Sec 3.12].)

The analysis of "new" G leads first to L standard in G (lec2);
then argues directly (i.e. not quoting std-form literature):
For $K := C_G(L)$, we obtain at [AS04d, p 1183]
a distinct G-conjugate R of K, with $N_R(K)$ of even order;
as otherwise, we get the suff-condition of [AS04d, I.8.2],
for $N_G(L)$ str-embedded (so G is "old" char-2: has no L).

Thereafter, most possible L are eliminated; but for $L \cong L_2(4)$,
get R of order 2 (indeed $\leq L$), so $C_G(z) \cong \mathbb{Z}_2 \times L_2(4)$;
and recognize G as "new" J_1 by Janko [AS04d, I.4.9].

An application to permutation groups

One version of strongly embedded (in Bender's original title):

- All involutions fix exactly one point of G/M.

Holt [ALSS11, B.2.1] (also F. Smith) extended the analysis to:

- *Some* 2-central involution fixes exactly one point;

where the only new groups to arise are S_n and A_n for odd n.

Exer 8.7: Check the 2-central condition in these new groups.

Holt reduces to a certain fusion condition at [Hol78, 4.1]:[50]
as failure would give Aschbacher's sufficient cond [Asc73]
for G to have a strongly embedded subgroup ("old" case).

The "new" cases arise toward the bottom of [Hol78, p 182].

For typical applications of this in CFSG, see e.g. [ALSS11].
Often toward the end of proofs, to force $H = G$
when a conclusion-group H might have odd index in G:
show G/H has Holt's hypothesis—but not his conclusion.

---

[50] Cf. my notes at `www.math.uic.edu/~smiths/notes.pdf` ; and Stroth's (thanks!) `www.math.uic.edu/~smiths/stroth.holt.pdf` , referring to [PS14] .

We'll PAUSE here; before moving on to versions for odd p...

Lecture 8a: Analogous fusion results for odd primes

(version of 25sept2015)

Stephen D. Smith
U. Illinois–Chicago

Venice Summer School
7-18 September 2015

Overview of the talk:

1: The Z_p^*-theorem for odd p
2: Thompson-style transfer for odd p
3: Strongly p-embedded subgroups for odd p

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

§1: The Z_p^*-theorem for odd p

The odd-p analogue of the Z^*-Theorem states:

If z of order p in H commutes with no distinct conjugate,
then $z \in Z_p^*(H)$—the preimage of $Z\big(H/O_{p'}(H)\big)$.

Here, unlike for $p = 2$, no "elementary" proof is known.

But experts knew early that it could be obtained using CFSG.
E.g. Artemovich published a proof in [Art88a];
and Guralnick-Robinson gave a sketch in [GR93, 4.1],
from which we extract a few features:

The hypothesis implies z is central in a Sylow p-group P of H.
Get P noncyclic using Frobenius Normal p-Complement Thm;
then reduce to z nontrivial on all (simple) components L.

Gross [Gro82] had studied the list of simple L,
to show p-central automorphisms are inner (p odd!).
(Recall lec1a Out(L); e.g. field automorphisms in Lie type...)
Exer 8.8: Check "inner" fails at $p = 2$—using transp of S_6.
This then leads to $z \in L$, with $H = L$ simple.

Gorenstein [Gor82, 4.250] had examined the list of L,
to show that just a few cases have the weaker condition:
that all L-fusion of z in P should fall inside $\langle z \rangle$.
But in those , z *is* fused to a nontrivial power z^i,
whereas our z commutes with no distinct conjugate. □

Waldecker's alternative proof [Wal13] for $p = 2$
proceeds as far as possible by local analysis;
but her known-centralizers hyp involves the CFSG-list.

We mention that in Guralnick-Robinson [GR93],
the Z_p^*-theorem arises in their wider setup—of generalizations of the Baer-Suzuki Theorem [GLS96, 15.5], which states:

If a p-element x has $\langle x, x^h \rangle$ a p-group for all $h \in H$, then $x \in O_p(H)$.

Robinson has also considered [Rob90][Rob09] approaches to proving the Z_p^*-theorem via p-block theory.

2: Thompson-style transfer for odd p

This time: the elementary proof of Thompson Transfer
does have suitable extension(s) to odd p.
(Sometimes called Thompson-Lyons transfer.)

One fairly general extension appears in [GLS96, 15.15];
the special case [GLS96, 15.17] is perhaps clearer:
Assume Q has index p in Sylow P of H, and z of order p has $z^H \cap P \subseteq zQ$. Then $z \notin O^p(H)$.

(We could weaken the hypothesis to *extremal* $x \in z^H \cap P$;
namely with $C_P(x)$ Sylow in $C_H(x)$.)

Lynd [Lyn14] extends the analysis to fusion systems.

My impression is that the case of odd p has not been
as influential in applications as original Thompson transfer.

§3: Strongly p-embedded subgroups for odd p

For odd p with P Sylow in G , M strongly p-embedded means
that $N_G(X) \leq M < G$ for all $1 < X \leq P$.

Again unlike $p = 2$,
for odd p, no independent determination of cases is known.

We saw (lec2) that the weaker *almost* strongly p-embedded
arises in the "Uniqueness" subcase of the Even Case.

So for use in their inductive situations,
Gorenstein and Lyons list [GL83, 24.1][GLS98, 7.6.1]
the cases for known-simple G (and P noncyclic).
(Hence for all-simple G—once CFSG was completed.)

The list contains the expected Lie-rank 1 groups in char p;
along with A_{2p} and a few cases for small p.

Of course the proof involves detailed examination
of the p-local structure of the simple groups.

Similar analysis arises in the revisionism-approach (lec2a)
to CFSG: see e.g. [GLS96, Sec 17] [GLS99, Ch 3].

Strong p-embedding also appears in the CFSG-approach of Meierfrankenfeld-Stellmacher-Stroth et al (lec2a); see e.g. Parker-Stroth [PS11].

Such groups have a sizable literature outside CFSG. A sample:

In p-modular representation theory, e.g.:

- Zhang [Zha94] deduces existence of a p-block of defect 0.
- Robinson [Rob11] et al study, for endotrivial modules.

Strongly p-embedded subgroups inside p-locals are especially important for *conjugation families*; e.g. an early discussion is in Miyamoto [Miy77].

These are related to applications in alg-topology contexts; e.g.:

- for saturated fusion systems—e.g. Oliver-Ventura [OV09];
- for rings of group invariants—e.g. Kemper [Kem01].

We mention that Brown [Bro00] considers strong p-embedding in the context of the probability of generating G (lec6a).

THANKS!

Lecture 9: Some more group-theoretic applications

(version of 27sept2015)

Stephen D. Smith
U. Illinois–Chicago

Venice Summer School
7-18 September 2015

Overview of the talk:

1: Distance-transitive graphs
2: The proportion of p-singular elements
3: Root subgroups of maximal tori in Lie-type groups

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

§1: Distance-transitive graphs

Various surveys for this active area are available.
We had mentioned (lec6) Li-Niemeyer-Praeger [PLN97].
Others include:

- Brouwer-Cohen-Neumaier [BCN89]
- Ivanov [Iva94]
- Cohen (in [BW04])
- van Bon [vB07]

We extract a few features of this now well-advanced project:

A group G is *distance-transitive* on a connected graph Γ
if for each i, G is transitive on pairs of vertices at distance i.
There are some standard classes of examples, including:
E.g. vertices of (hyper)cubes (among Hamming graphs);
(**Exer 9.1:** Check cubic case(s)—using diagonals...)
+Johnson,Grassmann,odd-graphs; some from forms/codes...
(See e.g. [BCN89] [PLN97, Sec 7.1] [vB07, Sec 2].)
But maybe not too many more? I.e. classify?

There is a process for reducing to the case of G primitive;
then, can apply the Aschbacher-O'Nan-Scott Thm (lec6).
(Which depends on CFSG via the Schreier Conjecture.)
Notably Praeger-Saxl-Yokoyama [PSY87] showed:
If G prim+dist-tr, diam(Γ) ≥ 3, then G has one of types:
PA (with Γ Hamming); or affine HA; or almost-simple AS.
The HA and AS cases motivated further use of the CFSG.

For affine HA with $F^*\big(\ G/O_r(G)\ \big)$ simple, various authors
(Liebeck-Praeger, van Bon, Ivanov, Saxl, Cohen ...)
treated the alternating, sporadic, and Lie-type cases,
with the final steps done by van Bon; see e.g. [vB07, Sec 5].

For almost-simple AS, again various authors
(Ivanov, Saxl, Liebeck, Praeger, van Bon, Cohen ...)
treated alternating, sporadic, and linear-group cases.
Partial results are available for some other Lie-type cases;
for the status, see e.g. [vB07, Sec 4].

Proofs involve detailed properties of the various simple groups,
for example maximal subgroups (lec6).
But the conclusion-graphs are comparatively rare;
and often require strong restrictions on various parameters,
so that arguments are often computational in nature.

§2: The proportion of p-singular elements

(Suggested by Cheryl Praeger and Bill Kantor)

Computational group theorists are interested in the efficiency of algorithms to search for p-singular elements.

A random search can rely on the proportion of such elements.

Isaacs-Kantor-Spaltenstein [IKS95] used CFSG to establish:

For p dividing the order of permutation group G of degree n, the proportion of p-singular elements is at least $\frac{1}{n}$.

(Equality iff $G = S_p$, or $n = p^a$ with G sharply 2-trans.)

Exer 9.2: Check the proportion in the S_p case.

Notice the main bound does not depend on p!

Section 2 of [IKS95] reduces to G almost-simple.

The main logic sequence roughly follows reduction to type AS in proving the Aschbacher-O'Nan-Scott Theorem (lec6).

We'll sample a little of the further argument:

Now the possible simple $L := F^*(G)$ must be examined.

Sec 3 indicates the comparatively easy calculations
for L alternating or sporadic; reducing to L of Lie type.

Sec 10 handles the somewhat easier case $p = \mathrm{char}(L)$,
where p-elements are unipotent (so use that theory).

Otherwise p-elements lie in some (typically non-split) torus.
Here the calculations use the minimal permutation degree of L;
these are tabulated in Section 4,
using existing estimates from the literature.
Tori are parametrized by conjugacy classes of Weyl group W;
with p-elements realized as block-diagonal matrices,
typically commuting with suitable unipotent elements.
Section 7 makes the calculations (with some computer use)
for exceptional L, type by type;
while Section 8 handles classical groups more uniformly.

Beyond the obvious use for p-element searching,
the result has been applied to computational problems
(for some given G) such as:

- random generation of G;
- recognition of G e.g. as a classical group;
- membership in G of an element of an overgroup of G.

(These can be followed via MathSciNet, starting at [IKS95].)

Niemeyer-Praeger [NP10] analyzed the context
and extended the methods.

§3: Root subgroups of maximal tori in Lie-type groups

For simple Lie alg $\mathcal{G}/\mathbb{C}$ (recall lec1),
action of a Cartan subalgebra $\mathcal{H}$ (maximal torus)
describes subspaces of $\mathcal{G}$ via root structure and Weyl gp W;
The associated Lie gp/$\mathbb{C}$ has Cartan subgp, root subgps etc;
and we saw analogous Cartan H etc in finite Lie-type G.
Of course for a subgroup X invariant under H,
the product XH is also a subgroup of G;
so these include much subgroup structure (cf. lec6) for G.

A basic work of Seitz [Sei83] extends from H-root groups
to T-root gps—for a maxl torus T (now typically non-split).
Exer 9.3: Give tori for e.g. $L_3(4)$. ($\leftrightarrow$ conj.cl. of $W = S_3$)
The work is mainly in the context of algebraic groups;
but subgroups of finite G might not display Lie structure,
so Seitz also made use of the CFSG for such situations.
We extract a few features:

To avoid complications in groups over small fields,
some results assume q and p "not too small"',
where G is defined over $\mathbb{F}_q$ for $q = p^a$.

The finite Lie-type group G arises as fixed points $\overline{G}^{\sigma}$,
for algebraic group $\overline{G}$ over $\overline{\mathbb{F}_p}$, under automorphism σ;
with $T = G \cap \overline{T}$, for a maximal torus $\overline{T}$ of $\overline{G}$.

The T-root groups are defined as the intersection of G
with groups generated by $\overline{T}$-root groups, in σ-orbits.

From (3.1), each is either a p-group (e.g. field automorphism)
or Lie-type over an extension field of $\mathbb{F}_q$ (e.g. graph aut).

Exer 9.4: Exhibit/compare these for $L_3(4)$ and $U_3(4)$.

Sketch: The T-root groups are used e.g. in (10.1) and (10.2),
to describe arbitrary overgroups Y of T in G. If thm fails?

Reduce at (10.11) to $F^*(G)$ at most 2 simple groups L.
Now apply CFSG for the list of such L; in cases alt,Lie,spor,
use structure of L to get numerical contradictions.

The results are used e.g. for subgroups of Lie-type groups,
notably in the study of maximal subgroups (lec6);
and indeed in other areas of applications of the CFSG, e.g.:

- generation and random walks (lec6a)
- fixed-point ratios (cf. lec10)
- model theory related to algebraic groups

We'll PAUSE here; before moving on to (briefer) applications...

Lecture 9a: More group applications (more briefly)
(version of 25sept2015)

Stephen D. Smith
U. Illinois–Chicago

Venice Summer School
7-18 September 2015

Overview of the talk:

1: Frobenius' conjecture on solutions of $x^n = 1$
2: Subgroups of prime-power index in simple groups
3: Application to 2-generation and module cohomology
4: Minimal nilpotent covers and solvability
5: Composition factors of permutation groups

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

§1: Frobenius' conjecture on solutions of $x^n = 1$

Frobenius (1895) conjectured:

If exactly n elements have $x^n = 1$ (for n dividing $|G|$),
then these elements give a subgroup of G.

(He had shown the number of solutions is a *multiple* of n.)

Exer 9.5: Explore: e.g. S_4 with $n = 8$; A_4 with $n = 4$; ...

Special cases were known (e.g. solvable by Hall [Hal76, 9.4.1]).
Zemlin [Zem54] reduced the problem to simple groups.
Iiyori and Yamaki announced the completed proof in [IY91],
having checked the types of simple groups in earlier papers.

How? When p divides both n and $|G|/n$, their Lm 1 shows:
either a p-Sylow is cyclic; or $p = 2$ and $m_2(G) \leq 2$.
(Here for $p = 2$ they examine the Small Odd list (lec2);
and for p odd, their Lemma 2 eliminates P.)

This reduces to n and $|G|/n$ coprime:
They illustrate some sample arguments in [IY91];
e.g. for Lie-type G, they use known structure of tori...

The result is relevant in the context of the *prime graph* of G,
with edges whenever G has an element of order pq;
There is a corresponding literature; see e.g. Hagie [Hag03].

§2: Subgroups of prime-power index in simple groups

For general groups H, there is no "co-Sylow" theory:
namely of subgroups K of p-power *index* in H.
(Solvable H have Hall p'-subgps (e.g. [ALSS11, A.1.14]).)

For simple G, the possible cases are given in Guralnick [Gur83].
The proof had essentially appeared in Liebler-Yellen [LY79],
though they only considered solvable K.
(Cf. also Kantor [Kan85] and Arad-Fisman [AF84].)
It is a comparatively easy deduction using the CFSG-list:

The cases G alternating or sporadic are quickly dealt with;
so consider G of Lie type.
If $\mathrm{char}(G) = p$, then $G = KU$ for a full unipotent group U;
so K is flag-transitive on the building,
where cases had been listed by Seitz [Sei73].
If not, WMA $K \geq U$; and then K is normal in a parabolic P;
and from order formulas, $|G : P|$ is rarely a prime power.

In particular one gets the permutation groups of prime degree.

The main result (sometimes via the prime-degree corollary) has been used in a wide variety of applications.

I won't attempt to give details here;
but a quick search on MathSciNet shows areas such as:

- maximal subgroups
- permutation groups
- ordinary and modular character theory
- Cayley graphs, distance-regular graphs
- group factorizations
- profinite groups
- Mersenne primes, Galois groups, group zeta functions
- codes, association schemes, game theory
- the Yang-Baxter equation in physics!

§3: Application: 2-generation and module cohomology

For cohomology $H^*(H, V)$ of H in a module V,
a number of results show (or else conjectures assert)
that H^* should be "not too large" in terms of V.

In particular Aschbacher-Guralnick [AG84, Thm A] show:
For finite H faithful on irreducible V over $\mathbb{F}_p$,
we have $|H^1(H, V)| < |V|$.

They reduce to H simple;
and consider H^1 via conjugacy of complements to V in HV;
Thm C gives generation of HV, wrt H^1 and generation of H.

Here they can apply [AG84, Thm B] for 2-generation.

That result (as mentioned in lec6a) proceeds via the CFSG.
The proofs for H alternating and Lie-type are fairly short;
the sporadic cases require a few more individual details.

Again these results are applied in many areas, such as:

- complements and module cohomology (e.g. H^2)
- generation and presentations
- permutation groups and Cayley graphs
- profinite groups

§4: Minimal nilpotent covers and solvability

Work of a number of authors has recently culminated in:
A group with a minimal nilpotent covering is solvable.
Here a covering is via union of a set of subgroups:
minimal if no subgroup can be removed;
and nilpotent if those subgroups are nilpotent.

Bryce-Serena [BS08b, 2.1] reduce to G almost-simple;
and when simple, they handle alternating and sporadic cases,
as well as several Lie-type cases.
Blyth-Fumagalli-Morigi [BFM15, Thm 2]
complete the analysis of the types of simple $F^*(G)$:
Lemma 2 easily reduces the Lie-type case to small rank.
Prop 6 eliminates outer-auts. So G simple; then finish
the 2 small-rank Lie-type cases remaining after [BS08b].

(The result is new; maybe still early for further applications...)

§5: Composition factors of permutation groups

(Suggested by Bill Kantor)

An important task in computational group theory is the determination of composition factors.
Luks [Luk87] gave a polynomial-time algorithm for finding composition factors in a permutation group.

The proof of correctness of the algorithm first shows the output H is a primitive group.
So cases are given in the Aschbacher-O'Nan-Scott Theorem. (Which (lec6) uses CFSG via the Schreier Conjecture.)
Further on p98, H is reduced to the almost-simple case AS; but also $H = H'$ there—so that again using Schreier, the output H is indeed simple (i.e. a composition factor).

The algorithm was later substantially improved e.g. [BLS97] (still using the Aschbacher-O'Nan-Scott Theorem);
and used in further problems such as conjugacy classes.

THANKS!

Lecture 10: Some applications farther afield
(version of 27sept2015)

Stephen D. Smith
U. Illinois–Chicago

Venice Summer School
7-18 September 2015

Overview of the talk:

1: Polynomial subgroup-growth in finitely generated groups
2: Relative Brauer groups of field extensions
3: Monodromy groups of covers of Riemann surfaces

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

§1: Polynomial subgroup-growth in fin.-gen. groups

In geometric and combinatorial group theory, relevant groups are typically infinite. (Finite *sections* can be relevant...)

But often they satisfy weaker finiteness conditions;
notably *finitely generated*; or
residually finite (intersection of finite-index subgps = 1); ...

The property of *subgroup-growth* is measured by expressing the number of subgroups of index n, as a function $f(n)$.

Of special interest is *polynomial* subgroup-growth (PSG) (rather than e.g. exponential).

Lubotzky-Mann-Segal in [LMS93] (building on earlier work) completed a characterization of such groups:

A finitely generated, residually finite G
has (PSG) $\Leftrightarrow$ it has finite rank and is virtually solvable.

Here *rank* r means: finite-index subgroups are r-generated;
and *virtually solvable*: some finite-index subgp is solvable.

(Solvable—suggests elim any finite simple groups/CFSG?)

The paper builds on (and provides an introduction to) Lubotzky-Mann [LM91] and Mann-Segal [MS90].

The CFSG is used at [LMS93, pp367ff], to identify (nonabelian) chief factors of finite quotients of G. (And similarly at [MS90, 4.1].)

(The proof uses the theory of infinite solvable groups...)
The Lemma there (similar to an argt of J.S. Wilson) shows that for N the centralizer of such chief factors, G/N must be a linear group in characteristic 0.
By earlier results, G/N with PSG is finite ext of solvable X/N.
Now any finite nonabelian chief factor of X would lie in a finite one of G—but ?such lie *above* X/N.
So X has any finite chief factors solvable; then an earlier[51] result leads to X itself, and then G, being finite/solvable. □

[51] I suspect "result ... in Section 3" there should be "Section 2"?

The result has inspired much further work
(e.g. analogues, such as polynomial "index growth");
and has been applied in many other areas, such as:

- profinite groups
- arithmetic groups and their zeta functions
- crystallographic groups
- branch groups of trees

A good survey article on the general area of growth in groups is Helfgott [Hel15].
(It is fairly explicit about uses/avoidance of CFSG.)

§2: Relative Brauer groups of field extensions

For a field K, the (abelian) *Brauer group* $B(K)$ is the set of:
Morita-equivalence classes (i.e. of module-categories)
of finite-rank central-simple algebras over K.

These are relevant to the classification of division algebras,
and to class field theory.

For an extension L/K, the *relative* Brauer group $B(L/K)$ is
the kernel of the natural map $B(K) \to B(L)$.

The term *global* field covers certain "1-generator" types:

- algebraic number fields (finite extension of $\mathbb{Q}$);
- function fields: of algebraic curves;
  or finite extensions of the rational functions $\mathbb{F}_q(t)$.

Note: We can realize a global extension L in the form $K(\alpha)$.

Fein-Kantor-Schacher [FKS81, Cor 4] showed:
For global fields $L > K$, the relative gp $B(L/K)$ is infinite.

We'll now extract some features:

This Cor 4 is deduced using their more general Thm 2,
which describes the p-part $B(L/K)_p$;
and more directly from its Corollary 3, essentially showing:
For E the Galois closure of L/K,
$B(L/K)_p$ is finite $\Leftrightarrow$ p-elements of $\mathrm{Gal}(E/K)$ fix roots
(of the minimal polynomial of α generating L/K).

Cor 4 now follows—using their group-theoretic Theorem 1:
For finite G transitive on set Ω of ≥ 2 points,
there is p, along with a p-element g fixing no points.
(This result is interesting for general permutation groups!)

If not: (*) each p dividing $|G|$ also divides $|G_\alpha|$.
Induction reduces to G simple(+primitive); so use CFSG-list.
For A_n, using (*) leads to G_α transitive on natl n-set of A_n;
finish using (*) with known number-theoretic estimates.
The Lie-type cases use (*), along with e.g.:
subgps of $L_2(q)$; long-root subgroups; Seitz flag-trans; ...
In sporadic cases, usually G_α normalizes a simple group,
and it is fairly easy to check that (*) can't then hold.

The results have inspired extensions etc in field theory;
and have applications in a number of other areas, including:

- elusive groups (no order-p element fixes a point)
- orbital partitions etc in permutation groups
- conjugacy class sizes in groups
- solvability criteria for groups
- factorizations in graph theory

Degrijse-Petrosyan [DP13] approach the global-field result via Bredon-Galois cohomology.

§3: Monodromy groups of covers of Riemann surfaces

We give a rough sketch of some algebraic geometry:

A *Riemann surface* is a 1-dim $\mathbb{C}$-manifold. (Take conn,cpct)
Its *genus* (say g) is the number of "handles".

A meromorphic $\phi : X \twoheadrightarrow$ the *Riemann sphere* (proj line/$\mathbb{C}$)
gives a *branched covering*. Removing the (finite) branch-pts
leads to a topological cover, with fundamental group etc.

Then lifting loops around those branch points
maps the (possibly huge?) fundamental group
to the finite *monodromy group* of the cover—which
often has cyclic/alternating composition factors; others?

Guralnick-Thompson in [GT90] conjectured:
For fixed g, only finitely many simple gps (beyond cyclic,A_n)
can arise as monodromy composition factors (over all X, ϕ).

Various authors then contributed to this project;
it was was completed by Frohardt-Magaard [FM01, Thm A].

We extract a few features:

Note by CFSG, as only 26 sporadics, "not alternating"
thus amounts to "finitely many Lie-type simple groups".
A standard re-formulation [GT90] is via perm gps H on Ω:
r generators x_i, with reln $x_1 x_2 \cdots x_r = 1$ (from fund gp); and
satisyfing $\sum_i \big(\, |\Omega| - \#(\text{orbits of } x_i) \, \big) = 2(\, |\Omega| + g - 1 \,)$.
To elim large H: get lower bounds on LHS, force $>$ RHS.
Can in fact proceed (Guralnick [Gur92]) via
upper bounds on *fixed-point ratio* $\frac{|\mathrm{Fix}_\Omega(x_i)|}{|\Omega|}$.

Sec 1 of [FM01] reviews the earlier history:
First [Gur92, 5.1ff] reduces from full monodromy H
to showing: only finitely many *almost-simple* subgroups H.
Then Liebeck-Saxl [LS91] treat large exceptional;
reducing to ... only finitely many *classical* $G := F^*(H)$.
Next Liebeck-Shalev [LS99b] treat "non-subspace" actions Ω;
reducing to Ω arising from subspaces of natl mod V for G.

Using fixed-point ratios for such V obtained in [FM00],
Frohardt-Magaard get group-theoretic Thm B; so Thm A.

The result has inspired a number of further refinements;
and applications in various directions, such as:

- orbits of group-elt tuples closed under braid operations
- non-simple abelian varieties

Fixed-point ratios are used in other areas;
e.g. Keller's [Kel05] alternative route to the $k(GV)$ problem.
(For p'-group G faithful on $\mathbb{F}_p$-module V,
show $\#$ (conj.cl. of GV) $\leq |V|$.)

See also e.g. Magaard-Waldecker [MW15b] [MW15a],
for some related applications of the CFSG.

We'll PAUSE here; before moving on to (briefer) applications...

Lecture 10a: Further exotic applications (more briefly)
(version of 25sept2015)

Stephen D. Smith
U. Illinois–Chicago

Venice Summer School
7-18 September 2015

Overview of the talk:

1: Locally finite simple Moufang loops
2: Waring's problem for simple groups
3: Expander graphs and approximate groups

longer handout: `www.math.uic.edu/~smiths/talkv.pdf`

§1: Locally finite simple Moufang loops

A *loop* has the group axioms, except perhaps associativity.
A *Moufang* loop has a weak associativity axiom:
$(ax)(ya) = a\big((xy)a\big)$.
Simple means a surjective homomorphism is an isomorphism.
Locally finite: finitely-generated substructures are finite.

Jon Hall [Hal07, Cor 1.3] showed:
A simple locally finite Moufang loop which is uncountable must in fact be a group (i.e. associative).

(1) Can *not* replace "uncountable" with finite or countable; as the example on the next page shows.
(2) Associativity is checkable locally, i.e. on triples; but seemingly uncountable-checking is crucial here?
(3) Even simplicity is "fairly" local; however, a Zaleskii example shows a locally-finite simple group need *not* be be limit of finitely-generated simple subgroups.

The story involves a standard non-group example,
namely the *Paige* loop $\mathrm{PSOct}(F)$:
the norm-1 (split) octonions (mod ± 1) over a field F.

In the finite case, Liebeck [Lie87b] obtained:
A finite simple Moufang loop is either a finite simple group,
or $\mathrm{PSOct}(F)$ for a finite field F.

He uses techniques of Glauberman and Doro related to triality;
and applies the CFSG to determine G with $S_3 \leq \mathrm{Out}(G)$.

Hall [Hal07, Thm 1.2] extended to the locally finite case:
A locally finite simple Moufang loop is
either a locally finite simple group,
or $\mathrm{PSOct}(F)$ for a locally finite field F.

The corollary now follows, because a locally finite field
is a subfield of countable $\overline{\mathbb{F}}_p$ for some p,
and the octonions are of dimension 8 over F (so countable).

Hall's is one of various later applications of Liebeck's result,
primarily in further developments in loop theory.

§2: Waring's problem for simple groups

(See e.g. [Tie14, Sec 3.3] and the survey [BGK14].)

Recall Lagrange: a positive integer is the sum of ≤ 4 squares.
The classical *Waring problem*, solved by Hilbert in 1909:
$\exists\ g(k)$: a pos integer is the sum of $\leq g(k)$ k-th powers.

We can ask the analogous question about finding $g(k)$,
for products of k-th powers in (simple) groups G.
Indeed $g(w)$, for a *word* w in G—other than just $w = x^k$.
(Ex: lec5a, Ore Conjecture; for $w = [x, y]$, got $g(w) = 1$.)
There is an increasing literature in this direction.

One strong result is that of Larsen-Shalev-Tiep [LST11]:
For any w there is N_w such that:
for $|G| > N_w$ we have $g(w) = 2$.
E.g. for large G, elements are products of 2 squares; cubes; ...
Of course the proof assumes the CFSG-list,
and uses detailed structure of the groups in that list.

§3: Expander graphs and approximate groups

The notion of *expander* graph arose in computer science;
for any reasonable subset S of vertices, and its neighbors ∂S,
the ratio $\frac{|\partial S|}{|S|}$ should be bounded above some $\epsilon > 0$.

Infinite increasing families, with fixed ϵ, have been built from
Cayley graphs of discrete subgroups in infinite Lie groups;
see e.g. Lubotzky [Lub94] for background.

Similar constructions have been applied
to families of finite Lie-type groups (e.g. increasing field).
(Again this assumes the CFSG-list and their properties.)

Indeed Kassabov-Lubotzky-Nikolov [KLN06] show
that for non-Suzuki families, this “always” works.

The Suzuki case was done by Breuilard-Green-Tao [BGT11];
they use Tao's notion of *k-approximate group*.
Roughly: a subset A of G, closed under inverses,
with products $A \cdot A$ covered by k translates gA.

For an introduction to approximate groups and their literature, see e.g Green [Gre12] and Breuillard [Bre14].

A few more uses of simple groups with approximate groups appear in Breuillard-Green-Tao [BGT12].

There are also further developments in the expander literature on constructions via simple groups.

THANKS!

for your attention, over these two very long weeks

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