**Math 215 - Homework**

Show all your work, written clearly in complete sentences.

1. Write the following sentence in symbols. You will need five propositions, which you can denote by L, M, S, G and P. State clearly what these propositions are.

If either labor or management is stubborn, then the strike will be settled if and only if the government obtains a court order, but police are not sent into the factory.

2. Let $P$ and $Q$ be the propositions "I wear a sweater" and "I brave the cold". Write (English) sentences which will mean the following. You may modify $P$ and $Q$ (e.g., you need not use "I"), but your sentences should make sense.

(a) $P \Rightarrow Q$

(b) $not\ Q \Rightarrow\ not\ P$

(c) $P$ is sufficient for $Q$

(d) $Q$ is necessary for $P$

3. p.53, no. 5

4. Write a truth table for $((P\ or\ Q)\ and\ (Q \Rightarrow\ not\ R))$. Your table should have 8 rows.

5. Let $a \in \mathbf{R}$ , let $P$ be the statement "$a > 5$" and let $Q$ be the statement "$a^2 + 2a - 35 > 0$". In each of the following statements give a proof or give a counterexample. Use the order axioms for $\mathbf{R}$ (p.24)

(a) $P$ implies $Q$

(b) $Q$ only if $P$

(c) $P$ is necessary for $Q$

(d) $P$ is sufficient for $Q$

(d) $P$ if and only if $Q$

6. Let P and Q be the statements "A function is differentiable" and "A function is continuous", respectively. Write in words:

(a) P implies Q

(b) The converse of "P implies Q"

(c) The contrapositive of "P implies Q"

(d) The converse of the contrapositive of "P implies Q"

7. State which of the following statements is a consequence of the statement: "If f and g are continuous at x=a and g(a) is not zero, then f/g is continuous at x=a."

(a) If f and g are continuous at x=a and $f(a) \neq 0$ then g/f is continuous at x=a.

(b) If f and g are continuous at x=a and g(a)=0, then f/g is not continuous at x=a.

(c) If f and g are continuous at x=a but f/g is not continuous at x=a, then g(a)=0.

(d) If f and f/g are continuous at x=a and $g(a) \neq 0$, then g is continuous at x=a.

8. Consider the statement: If $0 < n < 4$ then $n^3 < -2n^2 + 24n$. (Here $n \in Z$.)

(a) Construct a "backwards proof" of the statement, as on p.28.

(b) Now write a direct proof of the statement.

9. p.54, 6, (iii) : If $a$ and $b$ are real numbers prove that $(-a)(-b) = ab$.

[You can assume 6(i) and 6(ii) which were proved in class.]

10. p.54, 7: Prove by contradiction: If $n^2$ is even then $n$ is even.

11. Show by induction that

$$1^3 + 2^3 + \ldots n^3 = (n(n+1)/2)^2$$

.

12. Show by induction on $n$: If $n$ is a positive integer,

$$\frac{d^n}{dx^n}(\ln x) = (-1)^{n+1}(n-1)!\ x^{-n}$$

You can assume that $\frac{d}{dx}(\ln x) = 1/x$ and the formula for $\frac{d}{dx}(x^n)$ for a positive integer $n$.

13. p.55, 17. Hint: Do part (i) first, and use it to do part (ii).

14. The problem of the prisoner. (Handout in class.)

15. p.115, 1 (iii)

16. Let $A = \{x \in \mathbf{R} \mid |x - a| < 1\}$, $B = \{x \in \mathbf{R} \mid a - 1 < x < a + 1\}$. Show that $A = B$.

[Recall that $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$.]

17. Describe the power set $\mathcal{P}(X)$ in the following cases:

(a) $X = \{4\}$; (b) $X = \emptyset$; (c) $X = \{\emptyset\}$; (d) $X = \{\{1, 2\}, \{3\}, 1\}$.

18. Let $U$ be a universal set and $A, B \subseteq U$. Using the fact that $A^{cc} = A$, show that the law $(A \cap B)^c = A^c \cup B^c$ implies $(A \cup B)^c = A^c \cap B^c$ , the other law.

19. p.117, 12

20. p.117, 11, parts (vi) through (x)

21. p.117, 13 (ii)

22. p.117, 14

23. Let $f : \mathbf{R} \to \mathbf{R}$ be a function and $a, b \in \mathbf{R}$. The rigorous definition of $\lim_{x\to a} f(x) = b$ is

$$\forall \epsilon > 0,\ \exists \delta > 0,\ P(\epsilon, \delta)$$

where

$$P(\epsilon, \delta) : \forall x \in \mathbf{R}, (0 < |x - a| < \delta) \Rightarrow (|f(x) - b| < \epsilon).$$

(a) Let $f(x) = 2x + 4$ if $x \neq 1$ and $f(1) = 0$. Prove that $\lim_{x\to 1} f(x) = 6$.

(b) Write a rigorous definition in symbols of "not $(\lim_{x\to a} f(x) = b)$" .

Using this definition show that $\lim_{x\to 0} f(x) = b$ is false for all $b \in \mathbf{R}$, where $f(x) = 1$ for $x \geq 0$ and $f(x) = -1$ for $x < 0$. [Draw a picture of this function.]

Homework for Monday March 2: Corrections to your test.

24, 25. P.117, 16 and 18.

26. Extra problem to be given in class on Monday March 2.

27. This exercise is about the inclusion-exclusion principle.

(a) Suppose that $A, B \subset U$, where U is a universal set, $|U| = 21$, $|A| = 8$, $|B| = 7$, and $|A \cap B| = 3$. Find $|A^c \cap B^c|$.

(b) Suppose that each tile in a collection of 22 is a square or a circle and is also red or green. Suppose further that there are 12 square tiles, 8 red ones, and 11 which are both square and green. Use the principle of inclusion-exclusion to determine:

(i) the number of tiles which are square or green;

(ii) the number of tiles which are circles and red;

(iii) the number of tiles which are circles or red.

28. (a) Show that $f : Z \to Z^+$ given by

$$f(n) = \begin{cases} 2n : n > 0 \\ -2n + 1 : n \leq 0 \end{cases}$$

is a bijection.

[You can assume that any integer can be written as $2q$ or $2q + 1$ for a unique integer $q$. This problem shows that though $Z^+$ is a proper subset of $Z$, there is a bijection between $Z^+$ and $Z$!]

(b) Find the inverse $f^{-1}$ of $f$, where $f^{-1} : Z^+ \to Z$.

29. p.184, 12.

30. Let $X = \{1, 2, 3\}$. The six bijections $X \to X$ can be written as $f_1 : 1 \to 1, 2 \to 2, 3 \to 3$, $f_2 : 1 \to 2, 2 \to 1, 3 \to 3$, $f_3 : 1 \to 1, 2 \to 3, 3 \to 2$, $f_4 : 1 \to 3, 2 \to 2, 3 \to 1$, $f_5 : 1 \to 2, 2 \to 3, 3 \to 1$, $f_6 : 1 \to 3, 2 \to 1, 3 \to 2$.

(a) For each $f$, find the least positive integer $n$ such that $f^n = f.f...f(n \text{ times}) = 1$.

(b) Find the inverse of each $f$.

31. Use the pigeonhole principle to prove the following:

Let $S$ be a subset of $\{1, 2, ..., 2n\}$ such that $|S| = n + 1$. Then $S$ contains a pair of consecutive integers.

Hint: Consider the function $f : S \to \{1, 2, ...n\}$ given by $f(m) = m/2$ if $m$ is even and $f(m) = (m + 1)/2$ if $m$ is odd.

32. For the following pair of integers $a$ and $b$ use the Euclidean Algorithm to find $d = gcd(a, b)$ .

$a = 11391$, $b = 5673$

33. p.183, 7.

34. p.186, 23

35. Committees of 6 are to be formed from a group of 10 individuals. Give the answer in integers, not as binomial coefficients.

(a) How many such committees are there?

(b) How many committees of 6 individuals can be formed from the 10 if two particular individuals are to be included on the committee?

(c) How many committees of 6 individuals can be formed from the 10 if two particular individuals are to be excluded from the committee?

(d) How many committees of 6 individuals can be formed from the 10 if one particular individual is to be excluded from the committee?

(e) Use parts (c) and (d) and the inclusion-exclusion principle to find the number of committees of 6 individuals which can be formed from the 10, given at least one of two particular individuals is to be excluded.

36. p.225, 1

37. p.185, 18

Hints for p.185, 18 will be discussed in class. This problem is not straightforward, so leave enough time!

38. p.156, 2.7: Do the case $n = 4k+1$. The answer is in the back of the text, but you must show all the steps as was done in class for the case $n = 4k$. You must do both parts of the question.

39. Using the Binomial Theorem and the definition of the derivative of a function $f(x)$ as $f'(x) = \lim_{h\to 0}(f(x+h) - f(x))/h)$,

prove that if $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

40. See Definition 5.4.2, p.49 for the definition of the Fibonacci numbers $u_n$. Show that

$$u_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \ldots \binom{n-m}{m}$$

where $m = [n/2]$, the greatest integer not exceeding $n/2$.

Hint: Use $u_{n+1} = u_n + u_{n-1}$ and strong induction. This will be discussed in class. This is not difficult, but needs time.