LOCAL REAL ANALYTICITY OF SOLUTIONS FOR SUMS OF SQUARES OF NON-LINEAR VECTOR FIELDS

DAVID S. TARTAKOFF AND LUISA ZANGHIRATI

Abstract. We show that all smooth solutions of model non-linear sums of squares of vector fields are locally real analytic.

1. Introduction

We consider sums of squares of non linear vector fields, that is equations such as

$$P(x,u,D)u = \sum_1^q X_j^2 u = f$$

with the new feature that $\{X_j\}$ may depend in their "coefficients" on the solution u. As a prime example of this class we consider the following case (for $r > 0$):

$$P_u(D)v := \Big((D_x)^2 + (x^r D_t)^2 + (x^r \tilde{h}(x,t,u)D_t)^2\Big)\, v, \ (x,t) \in \mathbb{R}^2 \tag{1.1}$$

with $\tilde{h}$ real valued and real analytic in its arguments.

We shall assume our solution u to be C^∞, since smoothness (starting from $C^{2+\alpha}$) follows from the arguments of Xu [7] which are based on the subelliptic estimate clearly satisfied by P_u and the paradifferential calculus of Bony [1].

2. Results

Theorem 1. *If f is real analytic near (x_0,t_0), then so is any smooth solution to (1.1).*

We remark that the problem is significant in its own right and also because it bears the same resemblance to general quasilinear subelliptic partial differential equations that the sums of squares of linear vector fields do to the subelliptic complexes and 'boundary Laplacians' arising

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from the $\overline{\partial}_b$ operator in several complex variables. In particular, the local real analytic hypoellipticity of those (in the linear case) with symplectic characteristic variety (roughly corresponding to $r = 1$ here), proved independently by Treves and Tartakoff in 1978 ([4], [5], [6]), propels one quite reasonably to ask the same question in the quasilinear setting, of which the type of operator under study here is a simple prototype. (NB - the vector fields arising from $\overline{\partial}_{(b)}$ correspond more directly to $\partial_t - y\partial_t$ and $\partial_y + x\partial_t$ than to $\partial_t, \partial_y, x\partial_t$ and $y\partial_t$ as separate vector fields; nonetheless the "Grushin-type" operators have always provided the most tractable models.)

3. Proof

Using standard arguments it is easy to prove the following *a priori* estimates: $\forall s \geq 0, u \in C^\infty$ and compact $\mathcal{U}, \exists C = C_{s,u,\mathcal{U}} : \forall\, v \in C_0^\infty(\mathcal{U})$,

$$\sum_1^3 \|X_i v\|_s^2 + \|v\|^2_{s+\frac{1}{r+1}} \leq C\big\{|(P_u v, v)_s| + \|v\|_s^2\big\} \text{ and} \tag{3.1}$$

$$\sum_{i,j=1}^3 \|X_i X_j v\|_s^2 + \sum_1^3 \|X_i v\|^2_{s+\frac{1}{r+1}} + \|v\|^2_{s+\frac{2}{r+1}} \leq C\big\{\|P_u v\|_s^2 + \|v\|_s^2\big\} \tag{3.2}$$

where $\|\cdot\|_s = \|\cdot\|_{H^s},\ P_u \equiv P(x, u, D),\ X_1 = D_t,\ X_2 = x^r D_t,\ X_3 = X_3^{(u)} = x^r \tilde{h}(t, x, u) D_t$, and C depends only on the first $s + 3$ derivatives of u.

However the estimate we will need uses the maximality (and arbitrary positivity) of (3.2) rather than its subellipticity: with $|||v|||_s$ defined as follows for s a positive integer,

$$|||v|||_s \equiv \sum_{|I|\leq 2} \|X^I v\|_s, \quad (|||v|||_{H^s(\mathcal{U})} \equiv \sum_{|I|\leq 2} \|X^I v\|_{H^s(\mathcal{U})}, \text{for } \mathcal{U} \text{ open}) \tag{3.3}$$

then for any K and with $X^I = X^{I_1} X^{I_2} \ldots X^{I_{|I|}}$, $\exists C_K : \forall v \in C_0^\infty(\mathcal{U})$,

$$|||v|||_2 + K \sum_{|I|\leq 1} \|X^I v\|_2 \leq C\|P_u v\|_2 + C_K \|v\|_0. \tag{3.4}$$

4. The general scheme

The general scheme, as always, will be to use the *a priori* estimate applied to functions $v = \varphi D^m u$ and then to bring φD^m to the left of P_u modulo errors which are handled inductively. Noting that the *a priori* estimate provides for maximal control (i.e. no loss of derivatives) in the D_x direction, we limit ourselves to estimating $\varphi D_t^m u$. φ will be a smooth localizing function, namely identically equal to one in a fixed open set $\mathcal{U}_0$, where we wish to prove the solution u is analytic, and supported in $\mathcal{U}_1$, the open set where the data are assumed to be real analytic. The localizing function $\varphi(x,t)$ may be taken to be of the form $\tilde{\varphi}(t)\tilde{\varphi}(x)$, and terms with derivatives on $\tilde{\varphi}(x)$ may be disregarded since the operator is elliptic when x is away from 0, namely in the support of derivatives of $\tilde{\varphi}(x)$. . Thus for our purposes, $\varphi = \varphi(t)$ alone.

Taking $Pu = 0$ without loss of generality, we have from (3.4):

$$\frac{|||\varphi D_t^m u|||_2}{m!} \sim \frac{\|X_j^2\varphi D_t^m u\|_2}{m!} + \ldots \leq \frac{\|P\varphi D_t^m u\|_2}{m!} + \ldots \leq \frac{\sum\|[X_j^2, \varphi D_t^m]u\|_2}{m!} + \ldots$$

$$\lesssim \sum_{k=1}^{2}\frac{\|g_k(x,t,u,u')\varphi^{(k)}x^{2r}D_t^{m+2-k}u\|_2}{m!} + C\frac{\|\varphi x^{2r}[h(u), D_t^m u]D_t^2 u\|_2}{m!} + \ldots$$

$$\lesssim C\sum_{k=1}^{2}\frac{\|\varphi^{(k)}X^2 D_t^{m-k}u\|_2}{m!} + C\frac{\|\varphi x^{2r}[h(u), D_t^m u]D_t^2 u\|_2}{m!} + \ldots \tag{4.1}$$

considering $x^{2r}D_t^2 = X^2$, writing $h(\cdot) \equiv \tilde{h}^2(\cdot)$ and estimating the norm $\|g_k(x,t,u,u')\|_{H^2(\mathcal{U}_1)}$ by a constant. Here the $g_k(x,t,u,u')$ stand for the coefficients, aside from x^{2r}, which enter when φ is differentiated once or twice, and the dots "..." denote terms arising from lower order terms in the operator P, terms containing fewer X's.

We focus on the bracket in the last norm, the crucial one. To expand $(D_t^{m'}h)(u(x,t))$, we will need to use the Faà di Bruno formula or rather, what will suffice, and probably be more transparent, crude bounds for the results: writing

$$D_t^m g(u(x,t)) = (u'D_u + D_t)^{m-1}u'D_u g(u(x,t));$$

with primes on u denoting t derivatives, writing this roughly as

$$D_t^m g(u(x,t)) = ((u'\sigma + D_t)^{m-1}u'_{|\sigma = D_g})g,$$

i.e., σ becomes a 'countem' for the number of derivatives received by g. Then this is *at worst*

$$\sum_{m'}\binom{m}{m'}g^{(m-m')}(D_t^{m'}u'^{m-m'}). \tag{4.2}$$

Finally, distributing α objects into β positions yields

$$\frac{D^a u'^b}{a!} = \sum_{a_1+\cdots+a_b=a}\frac{u'^{(a_1)}}{a_1!}\cdots\frac{u'^{(a_b)}}{a_b!}$$

Thus we have

$$\frac{[D_t^m, P]}{m!} = \sum_{m'=1}^{m} \frac{(D_t^{m'} h(u(x,t)))}{m'!}\,\frac{(x^{2r} D_t^2) D_t^{m-m'}}{(m-m')!}. \tag{4.3}$$

with (cf. (4.2)):

$$\frac{(D_t^{m'}) h(u(x,t))}{m'!} \sim \sum_{m'-m''\geq 1}^{m} \frac{h^{(m'-m'')}}{(m'-m'')!}\,\frac{(D_t^{m''}(u'^{m'-m''}))}{m''!}$$

$$= \sum_{m'-m''\geq 1}^{m} \frac{h^{(m'-m'')}}{(m'-m'')!} \sum_{\sum_1^{m'-m''} m_j''=m''} \frac{D_t^{m_1''}u'}{m_1''!} \cdots \frac{D_t^{m''_{m'-m''}}u'}{m''_{m'-m''}!}$$

or in all, with (4.1),

(4.4)

$$\frac{|||\varphi D_t^m u|||_2}{m!} \lesssim \frac{\|[\varphi D_t^m, P]u\|}{m!} + \ldots \lesssim \sum_{k=1}^{2} \frac{\|\varphi^{(k)} X^2 D_t^{m-k} u\|_2}{m!} + \ldots$$

$$+ \sum \left\| \varphi \, \frac{h^{(m'-m'')}}{(m'-m'')!} \frac{D_t^{m_1''}u'}{m_1''!} \cdots \frac{D_t^{m''_{m'-m''}}u'}{m''_{m'-m''}!}\, \frac{(x^{2r} D_t^2) D_t^{m-m'} u}{(m-m')!} \right\|_2$$

where the sum is over $m \geq m' - m'' \geq 1, \sum_{j=1}^{m'-m''} m_j'' = m''$ and so $\sum_{j=1}^{m'-m''} (m_j'' + 1) = m'$.

By associating x^{2r} with a different term in the product if necessary, we may assume that the last term is of greatest order, and hence that the others are of order *at most* $m/2$.

5. REMARKS ON THE LAST SUM

Several remarks are in order concerning the last right hand side.

First of all, in utilizing the property that H^2 is an algebra to take the product of norms, there will occur a constant raised to the $m' - m''$. But this is allowable, since there are $m' - m''$ derivatives on the (analytic) function h where we expect a constant to that power.

Secondly, that power that power always corresponds to the increase in number of terms of the form $D_t^{m_j''} u'/m_j''!$ inside the norm; in the end the number of these terms cannot exceed m, hence the constant cannot exceed C^m.

Thirdly, we will associate the localizing function φ, with the highest order term and take it out of the norm, introducing another one which is closely related to the number of derivatives in that term - in this case $m - m'$. In bringing φ out of the norm there may be one or two derivatives (or three or four, given the first terms on the right of (4.4)), and while they will presumably balance quite well with $m!$ we need to be sure that they balance as well with $(m-m')!$ when $m-m'$ may actually be rather small (a large drop may have occurred all at once). To this end we make the following observation: as m drops from m to $m-m'$, there have appeared $m'-m''$ *new* lower order terms, or $m'-m''+1$ terms of no greater order, counting the principal one. Thus we have

$$(m'-m''+1)(m-m') \geq m; \qquad \textit{i.e.}, \frac{m}{m-m'} \leq m'-m''+1, \tag{5.1}$$

the same factor that occurred before, and appears in the number of derivatives on h. Thus, again, we can afford $(m/m-m')^4$ without danger.

The fourth observation concerns the effect of the sum. The sum corresponds *at most* to the number of ways to partition m derivatives among at most m functions, generally many fewer. Denoting by D a derivative (m of them) and by u a copy of u (t of them) we are faced with the number of ways to 'identify' or select t items (the $u's$) from among $m+t$ items (the D's and u's) with the understanding that in an expression such as

$$\underbrace{\underbrace{DDDDD}_{m_1}u\underbrace{DDDDD}_{m_2}u\underbrace{DDDDD}_{m_3}u\underbrace{DDDDD}_{m_4}u\cdots\underbrace{DDDDD}_{m_t}u}_{m\ D's\text{ and }t(\leq m)\ u's} \tag{5.2}$$

the D's differentiate only the first u following. The answer is that there are certainly not more than $\binom{m+t}{t} \leq 2^{m+t} \leq 2^{2m} = 4^m$ ways. And while we have written this out only for the first complete iteration of the *a priori* estimate, it is a remarkable fact that the form of the sum does not change after multiple passes, and hence the number of terms involved is subject to the same bounds. What is more, the same analysis applies after iteration of (4.4) (cf. below) and thus the sum will also not pose a difficulty in proving analyticity and may be replaced by a supremum below.

Finally, when these considerations enter and readability is an issue due to the length of lines, we shall tacitly replace the sum by a supremum and omit a constant such as $C^{m'-m''+2}$.

6. THE LOCALIZING FUNCTIONS AND INTRODUCING NEW ONES

Proposition 1. *For any two open sets* $\Omega_0 \Subset \Omega_1$, *with separation* $d = \textit{dist.}(\Omega_0, \Omega_1^c)$ *and any natural number* N, *there exists a universal constant* C *depending only on the dimension and a function* $\Psi = \Psi_{\Omega_0,\Omega_1,N} \in$

$C_0^\infty(\Omega_1), \Psi \equiv 1$ *on* Ω_0 *with*

$$|D^\beta \Psi| \leq \left(\frac{C}{d}\right)^{|\beta|+1} N^{|\beta|}, \qquad |\beta| \leq 2N, \tag{6.1}$$

The first localizing function, $\varphi = \Psi_m$, satisfies:

$$\Psi_m \equiv 1 \text{ on } \mathcal{U}_0 \;, \Psi_m \in C_0^\infty(\mathcal{U}_{1/m}) \;,\; |\Psi_m^{(k)}| \leq c^k m^k, \; k \leq 4, \tag{6.2}$$

where we have set, for $a \geq 0$:

$$\mathcal{U}_a = \{(x,t) \in \mathcal{U}_1 \; : \operatorname{dist}((x,t), \mathcal{U}_0) < a(\operatorname{dist}(\mathcal{U}_0, \mathcal{U}_1^c)\} \;. \tag{6.3}$$

When the first localizing function needs to be replaced but, say, $\tilde{m}$ derivatives of u remain to be estimated, we shall localize it with a function identically equal to one on $\mathcal{U}_{1/m}$, the support of Ψ_m but dropping to zero in a band of width $1/\tilde{m}$ of the remaining distance $(a(1-1/m))$ to the complement of $\mathcal{U}_1$, i.e., supported in

$$\mathcal{U}_{\frac{1}{m}+(\frac{1}{\tilde{m}})(1-\frac{1}{m})} = \mathcal{U}_{\frac{1}{m}+\frac{m-1}{m\tilde{m}}} = \mathcal{U}_{1-(1-\frac{1}{m})(1-\frac{1}{\tilde{m}})}. \tag{6.4}$$

We shall denote such a function by ${}_{\frac{1}{m}}\Psi_{\tilde{m}}$ That is, ${}_\rho\Psi_\sigma$ satisfies:

$${}_\rho\Psi_\sigma \equiv 1 \text{ on } \mathcal{U}_\rho, \qquad {}_\rho\Psi_\sigma \in C_0^\infty(\mathcal{U}_{\rho+\frac{1}{\sigma}(1-\rho)} \Subset \mathcal{U}_1). \tag{6.5}$$

Derivatives of ${}_\rho\Psi_\sigma$ satisfy, with universal constant C:

$$|D^k \left({}_\rho\Psi_\sigma\right)| \leq C^k \left(\frac{\sigma}{1-\rho}\right)^k, \; k \leq 4. \tag{6.6}$$

uniformly in ρ, σ. Of course any other (fixed) bound for k would do.

While it is true that we could just write $\|\varphi w\|_s \leq c\|\varphi\|_s\|w\|_s$, for $s \geq 2$, to do so would incur at least two derivatives on φ with no gain on w. To avoid this difficulty, we use the following finer estimates of the H^2 norm of product of functions.

Proposition 2. *If* $\varphi, \tilde{\varphi}$ *are two smooth, compactly supported functions with* $\tilde{\varphi} \equiv 1$ *on supp* φ *then for every* $p \geq 2$

$$\|\varphi D^p u\|_2 \leq C^2 \sup_{q \leq 2} \|D^q \varphi\|_{L^\infty} \|\tilde{\varphi} D^{p-q} u\|_2 \;, \quad \textit{and} \tag{6.7}$$

$$\|\varphi D^p u\|_2 \leq C^2 \sup_{q \leq 2} \|D^q \varphi\|_{L^\infty} \|D^{p-q} u\|_{H^2(supp\ \varphi)} \;. \tag{6.8}$$

7. Expanding the norm of the product in (4.4)

The norm of the product in (4.4) will be replaced, as announced, by the product of the H^2 norms, most of which will have as new functions ${}_{1/m}\Psi_{m''_j}$: multiplying through by m,

$$
\begin{aligned}
(7.1)\quad \frac{|||{}_0\Psi_m D_t^m u|||_2}{(m-1)!} &\lesssim \sum_{k=1}^{2} \frac{\|{}_0\Psi_m^{(k)} X^2 D_t^{m-k} u\|_2}{(m-1)!} + \dots \\
+ \sup_{\substack{m \geq m'-m'' \geq 1 \\ \Sigma_{j=1}^{m'-m''} m''_j = m'' \\ (\Sigma_{j=1}^{m'-m''} (m''_j+1) = m')}} & \left(\prod_{j=1}^{m'-m''} \frac{C\|{}_{1/m}\Psi_{m''_j} D_t^{m''_j} u'\|_2}{m''_j!} \right) \frac{\|{}_0\Psi_m \, X^2 D_t^{m-m'} u\|_2}{(m-m'-1)!}
\end{aligned}
$$

where, using (5.1), the factor $m/m{-}m'$ which entered on the right from multiplying through by m and decreasing the last denominator by one is absorbed in a slightly larger constant $C^{m'-m''}$ in the product. We have also bounded the terms $\|h^{(r)}(x,t,u)/r!\|$ by C^r and distributed these constants, one per term in the product of norms of derivatives of u'.

To unify these two types of terms we could combine them into one sum, over $k+m' \geq 1$, but there is nothing new introduced by considering the couple of extra derivatives which the localizing functions may receive - there is compensation with decrease in m and we have already seen this effect - it is essentially one familiar in elliptic regularity proofs by L^2 methods, so we will omit the terms with $k > 0$.

Now we have seen that we may bring the last localizing function, ${}_0\Psi_m^{(k)}$, out of the last norm and introduce the next function, ${}_{1/m}\Psi_{m-m'}$, identically equal to one on the support of ${}_0\Psi_m$, with a larger constant C_h. According to the above Proposition, when bringing a localizing function out of the norm its L^∞ norm will contribute up to two or, if already differentiated, perhaps four factors of m with corresponding decrease in the number of derivatives on u. This disturbs the balance between number of derivatives and the factorial, but (5.1) shows that even factors of roughly $(m/m{-}m)^4$ merely serve to modify the constant C_h; we conclude that we may pass from one localizing function to the next without problems.

That is, applying (7.1) to its own last term, with m replaced by $m-m'$, and ignoring $k > 0$ for simplicity, we have, denoting by $\frac{1}{m_2} = 1-(1-\frac{1}{m})(1-\frac{1}{m-m'})$ the band used up by the supports of the first two localizing functions, which will depend on the choice of m', and once again ignoring the first term on the right,

$$\frac{|||_{1/m}\Psi_{m-m'}D_t^{m-m'}u|||_2}{(m-m'-1)!} \lesssim \tag{7.2}$$
$$+\sup\left(\prod_{j=1}^{\rho'-\rho''}\frac{C\|_{1/m_2}\Psi_{\rho''_j}D_t^{\rho''_j}u'\|_2}{\rho''_j!}\right)\frac{\|_{1/m}\Psi_{m-m'}\,X^2D_t^{m-m'-\rho'}u\|_2}{(m-m'-\rho'-1)!}$$

or together,

$$\frac{|||_0\Psi_mD_t^mu|||_2}{(m-1)!} \lesssim \sup\left(\prod_{j=1}^{m'-m''}\frac{C\|_{1/m}\Psi_{m''_j}D_t^{m''_j}u'\|_2}{m''_j!}\right)\times \tag{7.3}$$
$$\times\left(\prod_{j=1}^{\rho'-\rho''}\frac{C\|_{1/m_2}\Psi_{\rho''_j}D_t^{\rho''_j}u'\|_2}{\rho''_j!}\right)\frac{\|_{1/m}\Psi_{m-m'}\,X^2D_t^{m-m'-\rho'}u\|_2}{(m-m'-\rho'-1)!}$$

where the supremum is over both sets of indices: $m \geq m'+\rho'-m''-\rho''$ and $\sum_{j=1}^{m'-m''}\sum_{k=1}^{\rho'-\rho''}(m''_j+\rho''_k) = m''+\rho''$ so if we set $s' = m'+\rho'$ and $s'' = m''+\rho''$, we have a sum over all $m \geq s'-s'' \geq 2$ and $\sum_{j+k=2}^{s'-s''} s''_{j+k} = s''$ while after the first iteration the sum was over all indices such that $m \geq m'-m'' \geq 1, \sum_{j=1}^{m'-m''} m''_j = m''$. In both cases, and for all succeeding ones, the number of such possibilities was seen by (5.2) to be bounded by C^m.

We continue this process, pulling the localizing function ${}_{1/m}\Psi_{m-m'}$ out of the last norm and replacing it with ${}_{1/m_2}\Psi_{m-m'-\rho'}$, subjecting that term to the *a priori* estimate, etc. Each time there is a whole 'spray' of far lower order terms, but the number of these is $s'-s''$, each has a suitable localizing function which will let us pass to a subsequent one by placing one (universal) constant with each new copy of u', and in the end we have a product of on the order of m terms of the form $\|D^ru\|_{H^2(\mathcal{U}_1)}$ all of order $r \leq 4$, say. (After all, localizing functions need not be introduced at the last stages - or even in any of the above, until we need to estimate a given term carefully - for instance, in the product in (7.1) the terms could easily have been left as $\frac{\|D_t^{m''_j}u'\|_{H^2(\mathcal{U}_{1/m})}}{m''_j!}$,) at least until the time came to subject that term to the *a priori* estimate to reduce its order (in case all other terms had been reduced to lower order).

We also need to remark at the end that what was true for the first localizing function, namely (5.1), will be a little different, since the next localizing function may bring not a factor of $m-m'$ with each derivative it receives but rather the factor (cf. (6.6))

$$\frac{m-m'}{1-\frac{1}{m}} = (m-m')\left(\frac{m}{m-1}\right)$$

so that, passing from $m-m'$ to $m-m'-n'$ we encounter instead of just

$$\frac{m}{m-m'} \leq m'-m''+1$$

an extra factor of $m/m-1$, possibly to the fourth power; and this may keep occurring as the order of the leading term keeps decreasing. For instance, after a few iterations, the analogous 'extra' factors from (6.6) will be

$$\left(\frac{m}{m-1}\right)\left(\frac{m-m_1}{m-m_1-1}\right)\left(\frac{m-m_1-m_2}{m-m_1-m_2-1}\right)\ldots$$

or even the fourth power of such a product. But there cannot be more than m terms in the product and each factor is far less than 2, leading to an easily acceptable constant C^m in the end.

This will prove the bounds for the laft hand side of (7.1)

$$\frac{|||_0\Psi_m D_t^m u|||_2}{(m-1)!} \leq C^{m+1}$$

uniformly in m and hence the analyticity of u in $\mathcal{U}_0$. □

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Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 851 So. Morgan St., Chicago IL 60607 USA
E-mail address: `dst@uic.edu`

Dipartimento di Matematica, Università di Ferrara, via Machiavelli 35, 44100 Ferrara, Italia.
E-mail address: `zan@dns.unife.it`