Math 310 – Summary
Charles Tier, Fall 2001

Text: Leon, *Linear Algebra with Applications, 6th ed.*

Chapter 1 - Matrices

Section 1:
- Linear systems
- Equivalent systems
- Triangular systems
- Back substitution
- Coefficient matrix
- Augmented matrix
- Elementary row operations I, II, III

Section 2:
- Row echelon form (REF)
- Lead and free variables
- Gaussian elimination
- Overdetermined system (more eqns than unknowns)
- Underdetermined system (more unknowns than eqns)
- Reduced row echelon form (RREF)
- Gauss-Jordan reduction
- Homogeneous equations (nontrivial solution is columns>rows)

Section 3:
- Scalars and matrices
- Vectors (row and column)
- R^n: Euclidean *n*-space
- Matrix arithmetic – equality, scalar multiply, addition, times, $AB \neq BA$
- Linear system – matrix formulation $A\mathbf{x}=\mathbf{b}$
- Linear combination
- Consistency Theorem for $A\mathbf{x}=\mathbf{b}$, $\mathbf{b}$ linear combination of columns
- Identity matrix $I_n = (\delta_{ij})$, $i=1,\ldots n$
- Inverse of a matrix (nonsingular, invertible)
- Singular – no inverse
- Transpose and basic rules: $(A^T)^T=A$, $(AB)^T=B^TA^T$, etc.
- Symmetric matrix: $A^T=A$

Section 4:
- Elementary matrices – relate to elementary row operations
- Row equivalent
- Key theorem on conditions for nonsingularity
- Diagonal and triangular matrices
- LU factorization

Chapter 2 - Determinants

Section 1:

Determinant of $n \times n$ matrix
Minor
Cofactor
Cofactor expansion to evaluate det(A)

Section 2:

Properties of determinants: $\det(A^T)=\det(A)$, $\det(A^{-1})=1/\det(A)$, etc.
Simplifying determinant using elementary row operations
A is singular iff det(A)=0

Section3:

Cramer's Rule, adjoint matrix, inverse using determinants

Chapter 3 – Vector Spaces

Section 1:

Euclidean vector space (review)
Vector space axioms and closure properties
C[a,b] and P_n
Properties of vector spaces

Section 2:

Definition of a subspace (closure and non-empty)
Null Space
Linear combination
Span (span $\rightarrow$ subspace)
Spanning set

Section 3:

Linear independence and dependence (geometric interpretation)
Relationship between linear dependence and singularity (Thm 3.3.1)

Section 4:

Basis – linearly independent and spanning set
Dimension – (# in basis) subspace $\{0\}$ has dim = 0
Standard basis

Section 5:

Review change of coordinates
Change of basis
Transition matrix (both directions)
General vector space changes (p. 170)

Section 6:

Row and column space
Two equivalent matrices have same row space
Rank
Consistency Thm for A**x**=**b, b in** row space
A $(n\times n)$ then A^{-1} exits iff columns are basis for R^n
Rank nullity thm – A $(m\times n)$ then rank + nullity = n
Column space properties –dim row space = dim col. space

Chapter 4 – Linear Transformations

Section 1:

Linear transformation or linear operator L:U $\rightarrow$ W
Special examples $L:R^2 \rightarrow R^2$, etc.
Kernel – ker(L) – subspace of V
Image of S: L(S)
Range: L(V) (subspace of W)

Section 2:

Matrix representation of linear trans. (tie to Chapters 1,2,3)
Construct matrix rep. in standard basis
Matrix representation thm. $L[v]_E = [w]_F$

Chapter 5 – Orthogonality

Section 1:

Scalar product $\mathbf{x}^T\mathbf{y} = \langle \mathbf{y}, \mathbf{x} \rangle$ (see inner product later)
Euclidean length $||\mathbf{x}||$
Relationship to angle
Unit vector
Orthogonal if $\langle \mathbf{x},\mathbf{y}\rangle = 0$ then $\mathbf{x} \perp \mathbf{y}$
Scalar and vector projections (important)

Section 2:

Defn of $X \perp Y$ if $\mathbf{x}\in X$ and $\mathbf{y}\in Y$ then $\langle \mathbf{x},\mathbf{y}\rangle=0$
Orthogonal complement
Fundamental subspaces $R(A)$, $R(A^T)$, $N(A)$, $N(A^T)$
Fundamental Thm of subspaces: $N(A^T)=R(A)^\perp$ and $N(A) = R(A^T)^\perp$
Direct sum
$(S^\perp)^\perp=S$
Fredholm Alternative Thm.(Cor. 5.2.5)

Section 3:

Solution of overdetermined systems (m>n)
Residual r(**x**) = **b**-A**x**
Least squares problem: A**z** = **p** where projection **p** in range closest to **b**

Normal equations: $A^TA\mathbf{z} = A^T\mathbf{b}$
Fit eqn. to data points

Section 4:
Defn. of inner product $< \mathbf{x}, \mathbf{y} >= \mathbf{y}^T\mathbf{x}$
Inner product space: V + <,>
Norm induced by <,>
Frobenius norm of matrix
Vector projection of x on y
Norm defn.
Normed linear space: V + || ||
$||x||_1$, $||x||_2$, $||x||_\infty$
norm and distance between vectors

Section 5:
Orthogonal set
Orthogonal set and linear independence
Orthonormal set (include functions)
Orthonormal basis representation thm. (Thm. 5.5.2)
Parseval's formula
Orthogonal matrix: $Q^{-1}=Q^T$, $||Qx||_2 = ||x||_2$
Projection of x onto subspace S (** important)
Approximation of functions

Section 6:
Gram-Schmidt process: construction and interpretation
QR factorization

Chapter 6 – Eigenvalues

Section 1:
Eigenvalues problem: $A\mathbf{x} = \lambda\mathbf{x}$
Eigenvalues
Eigenvectors and eigenspace – $N(A-\lambda I)$
Characteristic eqn. and polynomial
Complex eigenvalues
Trace
Sum and product of eigenvalues
Similar matrices

Section 2:
Solution of linear system of ODEs and tank problem
General solution
Initial value problem (IVP)
Complex roots case – general solution

Section 3:

e-vectors corresponding to distinct e-values are lin. indep.

Diagonalizable (similar to diagonal matrix)

$D = X^{-1} A X$ (X matrix of n lin. indep. e-vectors)

$A^n = X^{-1} D^n X$

Markov chain examples – long time behavior (eigenvector corresponding to eigenvalue 1)

Section 4: Hermitian and symmetric matrices

complex inner product and norm (see box on page 357)

A^H

A – real symmetric matrix

- eigenvalues are real
- eigenvectors corresponding to distinct e-values are orthogonal

unitary matrix (orthogonal for real symmetric matrix with rank *n*)

Schur's Thm

Spectral Thm: A – real symmetric then there exists orthogonal matrix U that diagonalizes A, $U^TAU = D$ (e-values need not be distinct ****)

normal matrix: $AA^T=A^TA$

A is normal iff A has complete set of orthonormal e-vectors