Math 481 Sample Exam Questions

1. Consider the function $f(x) = \pi - x$ on $0 < x < \pi$.

   (a) Sketch $f(x)$.

   (b) Find its Fourier sine series and sketch it on the interval $-2\pi < x < 2\pi$. Show all points of convergence.

   (c) Find its Fourier cosine series and sketch it on the interval $-2\pi < x < 2\pi$. Show all points of convergence.

   (d) Which Fourier series is more accurate over the entire interval?

2. Solve the heat conduction problem

$$u_t = u_{xx}, \quad 0 < x < L, \ t > 0$$
$$u_x(0,t) = 0 = u_x(L,t)$$
$$u(x,0) = 5 - 2\cos\tfrac{3\pi x}{L}$$

3. Determine the non-negative eigenvalues, if any, and their corresponding eigenfunctions for the BVP:

$$\phi'' + \lambda\phi = 0$$
$$\phi(0) = 0, \quad \phi'(\pi) = 0$$

4. Determine all the negative eigenvalues of:

$$\phi'' + 5\phi = -\lambda\phi$$
$$\phi(0) = 0, \quad \phi(\pi) = 0$$

5. Construct the solution of the wave equation

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < \pi, \ t > 0$$
$$u(0,t) = 0 = u(\pi,t)$$
$$u(x,0) = \sin x + 4\sin 3x, \quad u_t(x,0) = 0$$

Sketch the standing waves that are present and discuss the presence of waves moving to the right and left.

6. Use the method of eigenfunction expansion to solve Laplace's equation: $\nabla^2 u = u_{xx} + u_{yy} = 0$ in the unit square subject to the boundary conditions:

$$u(0, y) = 0, \quad u(1, y) = f(y), \quad u(x, 0) = 0, \quad u(x, 1) = 0.$$

7. Consider Laplace's equation $\nabla^2 u = u_{xx} + u_{yy} = 0$ inside the circular annulus $(1 < x^2 + y^2 < 4)$ subject to the boundary conditions

$$u = 0 \text{ on } x^2 + y^2 = 1, \quad u = G(\theta) \text{ on } x^2 + y^2 = 4.$$

   (a) Formulate the problem in polar coordinates and state all auxiliary conditions.

   (b) Use separation of variables to construct the solution.

8. Construct the Fourier series on $-L \le x \le L$ of

$$f(x) = \begin{cases} 1 & -L < x < 0, \\ 2 & 0 < x < L. \end{cases}$$

Sketch the Fourier series.