RESEARCH STATEMENT

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In both practical and scientific respects, it is of great interest to predict steady, periodic, quasi-periodic and fully developed turbulent characteristics of large-scale atmospheric and oceanic flows. Stability/bifurcation theory enables one to determine how different flow regimes appear and disappear as control parameters vary. It therefore provides one with a powerful tool to explore the predictive power of various models. My research in this direction is to study the structure robustness, stability and transitions of incompressible fluid flows and geophysical fluid flows. This study makes use of a nonlinear dynamics theory for the underlying physical problems consisting of the following three aspects:

1) complete bifurcation when the system parameter crosses some critical values,
2) asymptotic stability of bifurcated solutions, and
3) the structure/patterns and their stability/transitions in the physical spaces.

The study is based on two new theories developed recently by Ma and Wang [11, 12]. The first two aspects of the study are related to applications of a new bifurcation theory for dynamical systems. The third aspect is related to geometric study of flows in physical space. Topics in this direction include structural stability and structural bifurcation of divergence-free vector fields, and the links between kinematics and dynamics.

The general philosophy is that we first derive general existence of bifurcation to attractors, and then we classify the bifurcated attractors to derive detailed dynamics including, for instance, stability of the bifurcated solutions. The main ingredients of the theory include the attractor bifurcation theory together with new strategies for the Lyapunov-Schmidt and center manifold reductions. To carry out applications to specific problems, a crucial component is the center manifold and/or Lyapunov-Schmidt reductions of the original partial differential equations. Using this machinery, I have made progress on several problems (see [4, 5, 6, 7, 8, 9]).

Convection Problems for Oceanic Models

As mentioned in [2], stratification and rotation are two important ingredients of oceanic flows. In collaboration with Tian Ma and Shouhong Wang, we studied two typical oceanic models having these two ingredients. In [6, 7], we studied doubly-diffusive equations,

$$\begin{cases} \dfrac{\partial U}{\partial t} = \sigma(\Delta U - \nabla p) + \sigma(\lambda T - \eta S)k - (U\cdot\nabla)U, \\ \dfrac{\partial T}{\partial t} = \Delta T + w - (U\cdot\nabla)T, \\ \dfrac{\partial S}{\partial t} = \tau\Delta S + w - (U\cdot\nabla)S, \\ divU = 0, \end{cases} \tag{1}$$

and, in [8], we investigated the stratified, rotating, Boussinesq equations

$$\begin{cases} \dfrac{\partial U}{\partial t} = \sigma(\Delta U - \nabla p) + \sigma\lambda \cdot T \cdot k - \dfrac{1}{Ro} k \times U - (U \cdot \nabla)U, \\ \dfrac{\partial T}{\partial t} = \Delta T + w - (U \cdot \nabla)T, \\ divU = 0, \end{cases} \tag{2}$$

where $U = (u, v, w)$ is the fluid velocity, T is the temperature, S is the solute concentration, p is the pressure, g is the gravity constant, and k is the unit vector in the upward vertical direction. Suitable boundary conditions are imposed on the top and the bottom, while in the horizontal directions, periodic conditions are considered. The parameters are the thermal Rayleigh number λ, the salinity Rayleigh number η, the Prandtl number σ, the Lewis number τ and the Rossby number Ro.

We examine different transition/instability regimes defined by these parameters with the aim of obtaining a better understanding of the different physical mechanisms involved in the onset of convection. In comparison to the Rayleigh-Bénard convection case, the steady linearized problem around the basic state for Problem (1) or (2) is nonsymmetric. This leads to a non-self adjoint eigenvalue problem, and consequently much more involved bifurcation and stability analysis. As a consequence, the bifurcation and the flow structure are much richer. The crux of the analysis is the reduction of the problem to the center manifold in the first unstable eigendirections, based on an approximation formula for the center manifold function. The key idea is to find the approximation of the reduction to certain order, truly obtaining to a "nondegenerate" system with higher order perturbations.

Results on Problem (1)

In [6], we show that for the two-dimensional case there are two different transition regimes: continuous and jump, delineated by the critical parameter

$$\eta_{c_1} = \frac{27}{4}\pi^4\tau^3(1-\tau^2)^{-1}. \tag{3}$$

The structure of the bifurcated solution is clearly characterized.

For the three-dimensional case, we derived in [7] a quadratic form in terms of the relevant physical parameters which delineates the bifurcation types and we also gave an estimate of the dimension of the bifurcated solutions. A complete asymptotic stability analysis is also obtained.

Recently, in collaboration with J. L. Bona, T. Ma and S. Wang, in [4], we analyzed the 2-D Hopf bifurcation problem for (1). A complete stability analysis of the Hopf bifurcation is provided for a physically relevant parameter regime.

Results on Problem (2)

We obtained two main results for this problem in [8]. The first is a rigorous and complete bifurcation and stability analysis near the first eigenvalue of the linearized problem. The second is the onset of a Hopf bifurcation, leading to the existence of periodic solutions of the model. We derive in particular two critical Rayleigh numbers R_{c_1} and R_{c_2}. Here, R_{c_1} is the first critical Rayleigh number, while R_{c_2} is the critical value leading to the onset of a Hopf bifurcation. Both parameters are explicitly given in terms of the physical parameters. The crucial issues here include 1) a complete understanding of the spectrum, 2) identification of the critical Rayleigh numbers, and, most importantly, 3) the verification of the Principle of Exchange of Stabilities near these critical Rayleigh numbers.

Bifurcation and Stability of Burgers' Type Equation

Burgers' type equation appears as a prototype of equations in fluid dynamics and in combustion [1]. In collaboration with Xiaoming Wang, in [5], we study the dynamics of the following Burgers' type equation for $x \in [0, 1]$:

$$\begin{cases} u_t = u_{xx} + \lambda u - \lambda u u_x, \\ u(x,0) = u_0(x), \end{cases} \tag{4}$$

where λ is a positive real parameter. In this case, the linearized eigenvalue problem around the basic state is symmetric, and its spectrum can be explicitly given. In the case of Dirichlet boundary conditions, we get a supercritical attractor bifurcation when the parameter λ crosses the first eigenvalue. In the case of periodic boundary condition, with the help of Krasnoselski's theorem and the translation invariance of the equation, we obtain a bifurcated S^1 attractor consisting of steady states.

Structural Stability for 2D Incompressible Flows with Symmetry Many physical situations display symmetry and the associated mathematical analysis must take account of this. In paper [9] joint with Jian-Guo Liu and Cheng Wang, we study the structure and its evolutions of incompressible flows with certain symmetries. In addition, we explore and verify numerically structural transitions of flows using a simplified model of Marsigli oceanic flow model.

Ongoing Research Plans

I have several projects currently undergoing.

• Recently, in collaboration with Professor Jerry L. Bona, I started a new line of research concerning water waves equations, including Boussinesq systems, BBM type equations and KdV type equations. The objective in this direction includes:

(1) the orbital stability analysis for the cnoidal wave (periodic travelling wave) solutions to KdV type equations,
(2) a nonlinear existence theory for the second-order correct Boussinesq systems (see [3]), and
(3) the comparison between the second-order correct Boussinesq systems and a generalized KdV equation.

The Boussinesq systems in our study are derived from the 2D Euler equation. In (2) and (3), we study water waves in an open channel under the assumption that the fluid is irrotational, inviscid and uniform in the cross-channel direction.

• In the direction of convection problems for oceanic models, I am now making progress on stability analysis of the Hopf bifurcations for the problem (2). On the other hand, I intend to study more realistic geophysical models. For example, it would be interesting both mathematically and scientifically to include the Coriolis force term in the double-diffusive equations. Another interesting problem is to include surface tension effects in the double-diffusive model. By varying the aspect ratio, I expect to see regimes in which surface tension plays a role.

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