MCS 261:01 Hour Exam 1 Solutions

Questions;

1. Writ down the negation of following statements (5 points each).

   (a) Every integer is divisible by a prime.

   (b) For any real number $x > 0$, there is an integer n such that $n > x$.

2. Define $f : \mathbf{N} \to \mathbf{N}$ by $f(x) = x^2 + x$. Determine with reasons whether or not f is one-to-one or onto. (20 points)

3. Define $\sim$ on an interval $[-1, 2)$ by $a \sim b$ if and only if $\lfloor a \rfloor = \lfloor b \rfloor$.

   1. Show that $\sim$ defines an equivalence on $[-1, 2)$. (10 points)

   2 Find all equivalence classes. (10 points)

4. (a) Simplify $(p \vee q) \vee [(q \vee \neg r) \wedge (p \vee r)]$. (15 points)

   (b) Determine whether the following is a valid logical argument. (15 points)

   If I wear a purple coat and don't wear blue shoes, then I wear red socks.
   I am wearing a purple coat.
   If I wear blue shoes or red socks, then I wear a green hat.

   I am wearing a green hat.

5. Suppose that $A \subset B$ and $C \subset B^c$. Show that $A \cap C = \emptyset$. (20 points)

Solutions

1. (a) *This is a homework problem.*

   There exists an integer that is not divisible by any prime.

   (b) *modified from a review problem*

   There exists a real number $\mathbf{x} > \mathbf{0}$, such that for any integer n, we have $\mathbf{n} \leq \mathbf{x}$.

2. *an example shown in the review lecture*

   Answer: f is 1-1, but not onto.

   Reason: I. Suppose that $x_1, x_2 \in \mathbf{N}$ and $x_1^2 + x_1 = x_2^2 + x_2$. Then we have $x_1^2 - x_2^2 + x_1 - x_2 = 0$. After factoring, we get $(x_1 - x_2)(x_1 + x_2 + 1) = 0$. There seem to be two possibility 1. $x_1 - x_2 = 0$ and 2. $x_1 + x_2 + 1 = 0$. But since $x_1, x_2 \in \mathbf{N}$, $x_1 + x_2 + 1 \geq 1 + 1 + 1 = 3 > 0$. Therefore, we conclude that $x_1 - x_2 = 0$ or $x_1 = x_2$.

   II. f is not onto, because there is no $x \in \mathbf{N}$ such that $x^2 + x = 1$. In fact, for any $x \in \mathbf{N}$, $x^2 + x \geq 1 + 1 = 2 > 1$. (You can also use quadratic formula to substitute last sentence).

3. (a) *from homework problem section 3.1 # 21*

   $\sim$ is reflexive because $\lfloor a \rfloor = \lfloor a \rfloor$ for all $a \in A$. $\sim$ is symmetric because if $a \sim b$, i.e. $\lfloor a \rfloor = \lfloor b \rfloor$, then $\lfloor b \rfloor = \lfloor a \rfloor$ or $b \sim a$. $\sim$ is transitive because if $a \sim b$ and $b \sim c$, i.e., $\lfloor a \rfloor = \lfloor b \rfloor$ and

$\lfloor b \rfloor = \lfloor c \rfloor$, then $\lfloor b \rfloor = \lfloor c \rfloor$ or $a \sim c$. Thus $\sim$ is reflexive, symmetric, and transitive, so it is an equivalence relation.

(b) $\lfloor 0 \rfloor = 0$ so $\overline{0}$ is the set of all real numbers with $\lfloor x \rfloor = 0$, that is, the interval $[0, 1)$. $\overline{-1}$ is the set of all real numbers with $\lfloor x \rfloor = \lfloor -1 \rfloor = -1$, that is, the interval $[-1, 0)$. $\overline{1}$ is the set of all real numbers with $\lfloor x \rfloor = 1$, that is, the interval $[1, 2)$. Since $[-1, 2) = \overline{-1} \cup \overline{0} \cup \overline{1}$, equivalence classes on $[-1, 2)$ are $\overline{-1}$, $\overline{0}$ and $\overline{1}$.

4. (a) See textbook page 24, problem 16.

   (b) see sample exam 3 (b)

5. Proof by contradiction.

   Suppose $A \cap C = \emptyset$ is false. Then there exists a $x \in A \cap C$. Since $A \subset B$ and $x \in A$, we have $x \in B$ (by the definition of subsets). On the other hand, since $C \subset B^c$ and $x \in C$, we have $x \in B^c$. Together it gives $x \in B \cap B^c$. However, B and B^c does not share any element. This contradiction finishes our proof that $A \cap C = \emptyset$.