MCS 261, MAY 3, 2002 Name: ____________________
FINAL EXAM

◦ There are nine (9) problems on this exam.
◦ Each problem is worth 20 points.
◦ Write your solutions in your exam book.
◦ Turn in this sheet along with your solutions.
◦ Show and explain all of your work.
◦ **An unjustified answer is not correct!**
◦ If you use the logical equivalences on the attached page, refer to them by number. Refer to the logical arguments by name.

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1. Use the Euclidean algorithm to find $\gcd(-392, 273)$. Show all of your steps.

   Solution. The steps of the algorithm are as follows.

$$\begin{aligned} -392 &= (-2)273 + 154 \\ 273 &= 1 \cdot 154 + 119 \\ 154 &= 1 \cdot 119 + 35 \\ 119 &= 3 \cdot 35 + 14 \\ 35 &= 2 \cdot 14 + 7 \\ 14 &= 2 \cdot 7 + 0 \end{aligned}$$

   $\gcd(-392, 173) = 7$, the last nonzero remainder.

2. Let $A = \{1, 2, 3\}$ and $B = \{2, 4\}$.

   (a) Find $A \cup B$, $A \cap B$, $A \backslash B$, $A \oplus B$, and $A \times B$.

   Solution. $A \cup B = \{1, 2, 3, 4\}$, the elements in either set. $A \cap B = \{2\}$, the elements in both sets. $A \backslash B = \{1, 3\}$, the elements that are in A but not in B. $A \oplus B = \{1, 3, 4\}$, the elements that are in one set but not both. $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$, all pairs of elements from the two sets.

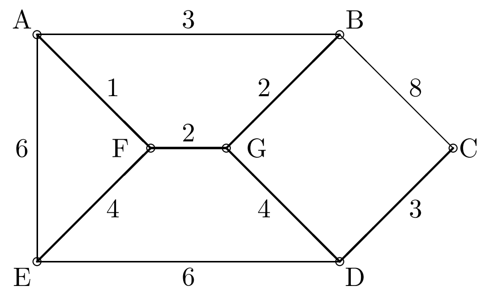
   (b) Let $\mathcal{R} = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$. Is $\mathcal{R}$ an equivalence relation on A? Why or why not?

   Solution. $\mathcal{R}$ is an equivalence relation. It is reflexive because for every $a \in A$, $(a, a) \in \mathcal{R}$. It is symmetric because if $(a, b) \in \mathcal{R}$, then $(b, a) \in \mathcal{R}$. It is transitive because if (a, b) and $(b, c) \in \mathcal{R}$, then $(a, c) \in \mathcal{R}$. A relation that is reflexive, symmetric, and transitive is an equivalence relation.

3. Find a minimum spanning tree of the following graph. You may draw your spanning tree on this sheet. In your exam book, state which algorithm you use and the order in which you add the edges to the tree.

   Solution. Using Kruskal's Algorithm, the edges are added in the following order: AF, FG, BG, CD, DG, EF. Using Prim's Algorithm starting with the vertex A, the edges are added in the following order: AF, FG, BG, DG, DC, EF.

Note: There may be some variation here when there are two or more possible edges at each point.



4. Let p and q be statements. Simplify the following expression completely.

$$[p \wedge (p \rightarrow q)] \rightarrow q$$

Solution. To simplify using the identities, use the following steps. Note that many of these steps can be combined.

$$\begin{aligned}
& & & [p \wedge (p \rightarrow q)] \rightarrow q \\
(13) & & \iff & \neg[p \wedge (\neg p \vee q)] \vee q \\
(6) & & \iff & [\neg p \vee \neg(\neg p \vee q)] \vee q \\
(6) & & \iff & [\neg p \vee (\neg(\neg p) \wedge \neg q)] \vee q \\
(5) & & \iff & [\neg p \vee (p \wedge \neg q)] \vee q \\
(4) & & \iff & [(\neg p \vee p) \wedge (\neg p \vee \neg q)] \vee q \\
(9) & & \iff & [\mathbf{1} \wedge (\neg p \vee \neg q)] \vee q \\
(7) & & \iff & (\neg p \vee \neg q) \vee q \\
(3) & & \iff & \neg p \vee (\neg q \vee q) \\
(9) & & \iff & \neg p \vee \mathbf{1} \\
(7) & & \iff & \mathbf{1}
\end{aligned}$$

Alternate Solution. The statement is equivalent to modus ponens, which is a valid logical argument so the statement is a tautology.

5. (a) Determine how many ways Sally can choose 5 of her 9 closest friends to invite over for dinner, assuming that Alice and Bob are married so if one is invited, they both must be invited. Be sure to explain how you get your answer.

   Solution. If Alice and Bob will be invited, there are $\binom{7}{3}$ ways to choose the other 3 from the 7 people remaining. If Alice and Bob are not invited, the 5 must be chosen from the other 7, which can be done in $\binom{7}{5}$ ways. Since these are disjoint cases, we add them to get the total number: $\binom{7}{3} + \binom{7}{5} = 35 + 21 = 56$ ways.

(b) Determine how many ways Sally and 5 of her friends can sit around her circular table, assuming that the group includes Charlie and Donna, who refuse to sit together. Be sure to explain how you get your answer.

Solution. We can count this by subtracting what we don't want. First, there is a total of 5! ways the 6 people can sit around the table. We want to subtract the number of ways that Charlie and Donna are sitting together. Put them together. There are 2 ways they can sit. Then the other 4 people are arranged to their right in 4! ways. So the number of ways they are together is $2 \times 4!$. Thus the number of ways they are not sitting together is $5! - 2 \times 4! = 120 - 2 \times 24 = 72$.

We can count this directly in the following way. First seat Charlie anywhere. Then Donna can't sit where Charlie is or on either side of him, so there are 3 possible places she can sit. The other 4 people are then arranged in the other 4 seats in 4! ways. So the total is $3 \times 4! = 3 \times 24 = 72$.

6. Let $f(n) = 3n^2$ and $g(n) = 2n^3$. Use the definition of Big Oh (not a theorem, property, or limit) to prove that $f = \mathcal{O}(g)$ but $g \neq \mathcal{O}(f)$.

Solution. First, we show that $f = \mathcal{O}(g)$. For $n \geq 1$, $n^2 \leq n^3$, so we have

$$|f(n)| = 3n^2 \leq 3n^3 = \frac{3}{2} \cdot 2n^3 = \frac{3}{2}|g(n)|,$$

so if we choose $n_0 = 1$ and $c = \frac{3}{2}$, we can say that $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$ so $f = \mathcal{O}(g)$.

Now we use contradiction to prove that $g \neq \mathcal{O}(f)$. Assume that $g = \mathcal{O}(f)$. Then there is an n_0 and a constant c such that $|g(n)| \leq c|f(n)|$ for all $n \geq n_0$, that is,

$$2n^3 \leq c \cdot 3n^2.$$

Dividing both sides by n^2 gives

$$\frac{2n^3}{3n^2} = \frac{2}{3}n \leq c.$$

This means that $\frac{2}{3}n$ is less than a constant for large n, which is a contradiction because $\frac{2}{3}n$ grows without bound. Thus we must have that $g \neq \mathcal{O}(f)$.

7. Recall that $3\mathbf{Z} + 1 = \{n \in \mathbf{Z} \mid n = 3k + 1 \text{ for some } k \in \mathbf{Z}\}$. Show that $\mathbf{Z}$ and $3\mathbf{Z} + 1$ have the same cardinality.

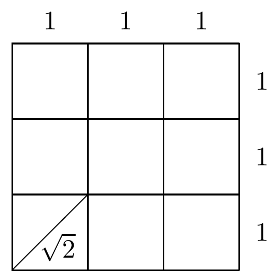
Solution. To show that two sets have the same cardinality we must show that there is a one-to-one correspondence between them. Define $f : \mathbf{Z} \to 3\mathbf{Z}+1$ by $f(n) = 3n + 1$. We will show that this is one-to-one and onto and is therefore a one-to-one correspondence.

To show f is one-to-one, we show that if $f(n_1) = f(n_2)$, then $n_1 = n_2$. Suppose $f(n_1) = f(n_2)$. Then $3n_1 + 1 = 3n_2 + 1$ so $3n_1 = 3n_2$ and $n_1 = n_2$ so it is one-to-one.

To show f is onto, we show that for every element of the range, there is an element of the domain that maps to it. Let $y \in 3\mathbf{Z} + 1$. Then by definition, $y = 3k + 1$ for some $k \in \mathbf{Z}$. Then $f(k) = 3k + 1 = y$. Thus for all $y \in 3\mathbf{Z} + 1$, there is a $k = \frac{y-1}{3}$ such that $f(k) = y$ so it is onto.

8. Ten points are chosen inside a square with side length 3. Prove that two of these points lie within $\sqrt{2}$ of each other.

   Solution. Divide the square into nine smaller squares with side lengths 1 as shown below. These 9 smaller squares will be the boxes and the 10 points will be the objects. By the Pigeon-Hole Principle, there must be some box with at least two objects, that is, one of the smaller squares has at least two of the points. The farthest apart two points can be and still lie in the same square is the length of a diagonal, which is $\sqrt{2}$. Thus by guaranteeing that two points are in the same small square, we guarantee that they are within $\sqrt{2}$ of each other.



9. A sequence is defined by the recursion relation

$$\begin{aligned} a_0 &= 1 \\ a_{n+1} &= 3a_n + 2^{n+1} \text{ for } n \geq 0. \end{aligned}$$

   Use mathematical induction to prove that

$$a_n = 3^{n+1} - 2^{n+1}$$

   for all $n \geq 0$. *Note:* You do not need strong induction for this proof.

   Solution. Base Case: $n = 0$. The formula says that $a_0 = 3^{0+1} - 2^{0+1} = 3 - 2 = 1$. This matches the definition, so the formula is true for $n = 0$.

   Inductive Step: Assume that $a_n = 3^{n+1} - 2^{n+1}$ and prove that $a_{n+1} = 3^{n+2} - 2^{n+2}$. By the definition and then using the induction hypothesis, we have

$$\begin{aligned} a_{n+1} &= 3a_n + 2^{n+1} \\ &= 3\left(3^{n+1} - 2^{n+1}\right) + 2^{n+1} \\ &= 3 \cdot 3^{n+1} - 3 \cdot 2^{n+1} + 2^{n+1} \\ &= 3^{n+2} - 2 \cdot 2^{n+1} \\ &= 3^{n+2} - 2^{n+2}. \end{aligned}$$

   Thus if it is true for n, then it is true for $n + 1$, so it is true for all $n \geq 0$.