Solution to the Final Exam - MCS 261

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Instructor: Keith E. Mellinger

1. (a) Prove or disprove the logical statement: $\neg(p \wedge q) \Leftrightarrow p \rightarrow \neg q$. You may use any valid method.
   Solution: $p \rightarrow \neg q \Leftrightarrow \neg p \vee \neg q \Leftrightarrow \neg(p \wedge q)$. The first equivalence comes from the negation of an implication. The second equivalence is DeMorgan's Law. So the two statements are equivalent.

   (b) State the negation of "If it rains, then I will not go to the park."
   Solution: It rains and I go to the park.

2. Let A be the set of integers and B be the set of rational numbers. Consider the function $f : A \rightarrow B$ defined by $f(n) = \frac{n}{3}$.

   (a) Is f one-to-one? Prove or disprove.
   Solution: Yes. Suppose $f(n_1) = f(n_2)$. Then, by definition of f, it follows that $\frac{n_1}{3} = \frac{n_2}{3}$. But this immediately implies that $n_1 = n_2$. So the function is one-to-one.

   (b) Is f onto? Prove or disprove.
   Solution: No. There is no integer n such that $f(n) = \frac{1}{2}$.

3. Prove that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

   for any $n \in \mathbb{N}$.

   Solution:

   1. **Base Case**: Let $n = 1$. The left hand side of the equation gives $1 \cdot 2 = 2$. The right hand side gives $\frac{1(2)(3)}{3} = 2$. Since these are equal, the statement holds for $n = 1$.

   2. **Inductive Hypothesis**: Assume that $\boxed{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}}$ for some $k \geq 1$.
   We now need to show that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$.

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + \cdots + (k+1)(k+2) &= 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= (k+1)(k+2)\left(\frac{k}{3} + 1\right) \\ &= \frac{(k+1)(k+2)(k+3)}{3}. \end{aligned}$$

   Therefore, the statement is true by mathematical induction.

4. Solve the recurrence relation

$$a_n = -5a_{n-1} + 6a_{n-2}, \quad n \geq 2,$$

given $a_0 = 5$ and $a_1 = 19$.

Solution: The characteristic polynomial is $x^2 + 5x - 6 = (x+6)(x-1)$. The characteristic roots are then $x = -6$ and $x = 1$. So the solution is $a_n = c_1(1^n) + c_2(-6)^n$ for some constants c_1 and c_2. Plugging in the initial conditions gives us

$$5 = a_0 = c_1 + c_2$$

$$19 = a_1 = c_1 + -6c_2.$$

The solutions to this system are $c_1 = 7$ and $c_2 = -2$. Our final solution is

$$a_n = 7 + -2(-6)^n \quad \text{for } n \geq 1.$$

5. Suppose $a, b, c \in \mathbb{Z}$ with $a \neq 0$ and $b \neq 0$. If $a|c$, $b|c$, and $\gcd(a, b) = 1$, prove that $ab|c$.

Solution: From the given conditions, we know that there exists integers w, x, y, z such that

- $aw = c$
- $bx = c$
- $ay + bz = 1$

We take the third equation and multiply both sides by c giving us $acy + bcz = c$. Now, we make some substitutions. For the "first" c, we substitute $c = bx$ and for the "second" c, we substitute $c = aw$. This gives us

$$a(bx)y + b(aw)z = c$$

which factors as

$$ab(xy + wz) = c.$$

This implies $ab|c$.

6. (a) Solve for x in the congruence: $3x \equiv 2 \pmod 5$.

   Solution: By trial and error, one quickly finds that $x = 4$. Recall that you only need to check values of x between 0 and 4.

   (b) Solve for x and y in the congruences

$$2x + y \equiv 5 \pmod 7$$

$$x - y \equiv 1 \pmod 7$$

   Solution: Adding the two congruences tells us that $3x \equiv 6 \pmod 7$ which implies that $x \equiv 2 \pmod 7$. Plugging back into the first congruence gives us $4 + y \equiv 5 \pmod 7$, so $y \equiv 1 \pmod 7$.

7. A bridal party consists of the bride, the groom, three bridesmaids, and three groomsmen (8 people total). After the wedding, they arrange themselves in a row for a picture.

   (a) How many arrangements are possible if the bride and groom must stand together (not necessarily in the middle)?

   Solution: There are 2 choices for how to arrange the bride and groom and then 7! ways to arrange the entire group. Answer: $2 \cdot 7!$

   (b) How many arrangements if, in addition to the bride and groom standing together, the men are all together and the women are all together?

   Solution: Again, there are 2 ways to arrange the bride and groom. This fixes the sides for the men and women. There are 3! ways to then arrange and men, and 3! ways to arrange the women. Answer: $2 \cdot 3! \cdot 3!$

   (c) After the pictures, how many ways are there for the bridal party to sit around a table if the bride and groom must sit together (men and woman not necessarily together)?

   Solution: We sit the groom down first. There are then 2 ways to decide where to put the bride and 6! ways to arrange the other members of the bridal party. Answer: $2 \cdot 6!$

8. A guitar club consists of 20 members, 7 women and 13 men including Keith and Dave. The club wants to send 5 of its members to a guitar contest.

   (a) How many ways to select who goes to the contest?

   Solution: $\binom{20}{5}$, since the order does not matter here.

   (b) How many ways to select who goes to the contest if Dave refuses to go with Keith?

   Solution: Either Dave goes and Keith doesn't, Keith goes and Dave doesn't, or neither of them goes. Answer: $\binom{18}{4} + \binom{18}{4} + \binom{18}{5}$

   (c) How many ways to select who goes to the contest if both men and women must go?

   Solution: I take the total from part (a) and subtract the number of ways to select all men or all women. Answer: $\binom{20}{5} - \binom{13}{5} - \binom{7}{5}$

9. (a) How many outcomes are possible from rolling three identical 12-sided dice? (*note: the dice are* ***identical***)

   Solution: We are selecting 3 outcomes from a collection of 12. The order does not matter (since the dice are identical) and we can select the same outcome more than once. Answer: $\binom{12+3-1}{3}$

   (b) How many ways are there to distribute 12 distinct candy bars to 3 children if each child should receive at least one candy bar? (*note: the candy bars are all* ***different***)

   Solution: The total number of ways to distribute the candy bars is 3^{12}, three choices for each candy bar. But, there are $3 \cdot 2^{12}$ ways to give the candy bars to only two of the kids, and there are 3 ways to give the candy bars to only one kid. By inclusion/exclusion, the answer is: $3^{12} - 3 \cdot 2^{12} + 3$.

   Many of you had trouble with this problem. Think about this: if you think of the candy bars as the "houses" and the kids as the "colors," this problem is exactly the same as the house coloring problem from the practice exam.

10. (a) Is the graph below Eulerian and/or Hamiltonian? Explain.

Solution: Recall that the graph is on 6 vertices labelled A - F. There are 9 edges: AE, AF, BC, BD, BF, CD, CE, DE, and EF. The graph is not Eulerian since there are 4 vertices of odd degree. By Euler's Theorem, a graph is Eulerian if and only if every vertex has even degree. The graph is Hamiltonian. A Hamiltonian cycle is given by the vertex sequence A, F, B, C, D, E, A.

(b) Draw a picture of all non-isomorphic trees of order 5. (*Note: a tree is a connected graph with no cycles*) *Solution:* First, recall that a graph is connected if between every pair of distinct vertices, there is a path. Hence, you need at least 4 edges in your graph in order for it to be connected. Keeping this in mind, there are 3 such graphs. The degree sequences are [4,1,1,1,1], [3,2,1,1,1], and [2,2,2,1,1],