SECTION 8.1 MARKOV PROCESSES

Students at a certain large university in a big Midwestern city use the library in measurable patterns. The day after a student goes to the library, the student will go again the next day 10% of the time, and will not go 90% of the time. On the day after a student did not go to the library, they will again not go 30% of the time, but will go to the library 70% of the time. According to library records, 40% of the students visited the library today.

a) Fill in the Transition Matrix below, using Probability values taken from the %-values described.
For example, students who went to the Library today are in a Current State L, and if they go again the next day, L again, 10% of the time, then 0.10 should be placed in the top left spot of the matrix.

Current State

$$\text{Next State} \quad A = \begin{array}{c} \\ L \\ NL \end{array} \begin{array}{c} \begin{array}{cc} L & NL \end{array} \\ \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{array}$$

Note that when properly filled in, EVERY column of the Transition Matrix will total 1.0, you should always double-check before proceeding. Such a matrix is called a Stochastic Matrix.

b) Now fill in the column matrix X_0 with the probabilities that a random student visited the library today.
Again, we note that the column has a total of 1.0

$$X_0 = \begin{array}{c} L \\ NL \end{array} \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

We use the general formula $X_{n+1} = AX_n$ to find "the next day's probabilities" from the current day.
This means we can use X_0 above to find X_1. In our formula, n = 0, so we are setting up this:

$X_{0+1} = AX_0$, and so $X_1 = AX_0$ Calculate those probabilities: $X_1 =$

c) What about the next day? Two days from today? We use $X_2 = X_{1+1} = AX_1$

Use your X_1 to find the probabilities in X_2:

d) Let's find another option. We noted that $X_2 = AX_1$, but we also saw that $X_1 = AX_0$, so if we place this equivalent of X_1 into $X_2 = AX_1$ getting the following conclusion: $X_2 = AX_1 = A(AX_0) = A^2 X_0$

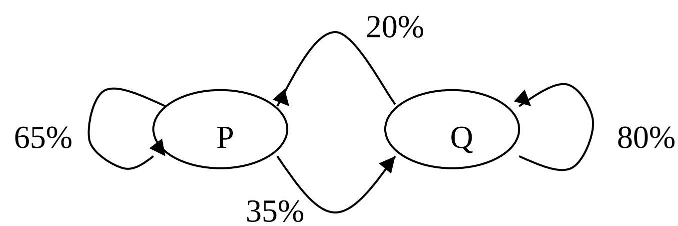
We can calculate X_2 directly from X_0. Use your calculator to do so, and verify it is the same probabilities as we found in part (c).

e) We can extend the idea further, using X_0 to find the probabilities any number of days into the future we wish to find. For example, if we want *n* days into the future, we can use $X_n = A^n X_0$

Use this to find the probabilities 5 days from today. Round to 3 decimal places, as necessary.

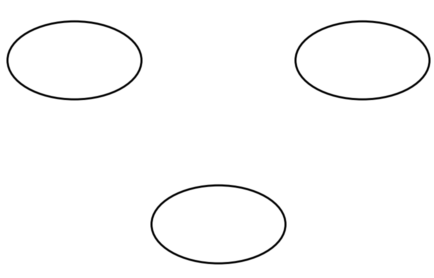
TRANSITION DIAGRAMS can be used to represent movement from one state to another, and for helping set up a proper Transition Matrix.

Build a fully labeled Transition Matrix, A



Now use this Transition Matrix to build a Transition Diagram:

$$A = \begin{array}{c} \\ E \\ F \\ G \end{array} \begin{array}{c} \begin{array}{ccc} E & F & G \end{array} \\ \begin{bmatrix} .6 & .3 & .15 \\ 0 & .5 & .75 \\ .4 & .2 & .1 \end{bmatrix} \end{array}$$



EXERCISE: Members of a local gym usage of the facility have been tracked by management. Members who use the gym for a long workout on a given day have been observed to have a long workout the next day 20% of the time, a short workout 50%, and none at all the rest of the time. For those who had a short workout on a given day, 60% will have a long workout the next day, 25% another short workout, and no workout the rest of the time. For those who did not come to the gym on a given day, half will have a long workout and half a short workout.

i) Build a Transition Diagram to represent the workout patterns of the members

ii) Build a Transition Matrix, fully labeled

iii) Suppose on Tuesday, 38% of the members come in for a long workout, 47% of the members come in for a short workout, and the other 15% do not come in that day. Build a Distribution Matrix to reflect Tuesday's probability distribution.

iv) Set up an appropriate calculation and determine what probability we expect on Wednesday for long, short, and no workouts by the gym members. Answer in %-form, rounded to the nearest 0.1

v) Set up an appropriate calculation and determine what probability we expect on Friday for long, short, and no workouts by the gym members. Answer in %-form, rounded to the nearest 0.1

vi) Set up an appropriate calculation and determine what probability we expect on Sunday for long, short, and no workouts by the gym members. Answer in %-form, rounded to the nearest 0.1

SECTION 8.2: MARKOV PROCESSES AND STEADY STATES

We revisit the university students and their library habits. Recall the Transition Matrix:

$$A = \begin{array}{c} \\ L \\ NL \end{array} \begin{array}{c} \begin{array}{cc} L & NL \end{array} \\ \begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix} \end{array}$$

Also, recall this was called a Stochastic Matrix. Because it also has no 0's in the matrix, we now note that it is a Regular Stochastic Matrix, which is a Stochastic Matrix where at least one power of A, A^n, has ALL non-Zero values. Check each of the following to see if they are Regular, raising them to whole number powers in your calculator, as necessary. Which are regular?

i) $\begin{bmatrix} 0 & .2 \\ 1 & .8 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix}$ iii) $\begin{bmatrix} .3 & 1 & 0 \\ .5 & 0 & 1 \\ .2 & 0 & 0 \end{bmatrix}$ iv) $\begin{bmatrix} 1 & .4 & 0 \\ 0 & .5 & 0 \\ 0 & .1 & 1 \end{bmatrix}$

Regular Stochastic Matrices allow for a "steady" probability occurring "in the long run". When enough transitions have occurred, our formula for finding the next time unit's probabilities, $X_{n+1} = AX_n$, becomes the more general $X = AX$, where the probabilities in X are the same before and after a transition. Returning to our library example, and using $X = \begin{bmatrix} x \\ y \end{bmatrix}$, set up the matrix equation $X = AX$ and turn it into a system of equations in x and y.

$$\begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} \quad \\ \quad \end{bmatrix} \quad \Rightarrow$$

Simplify the questions and note that they end up being identical. We then add an extra equation based on the fact that $x + y = 1$ is always true. Now solve the system once this is paired up to your above equation.

Your calculator can be used to verify these "steady probabilities". Calculate: $A^n X$ using a sufficiently large value for *n*.

Now that we have steady state probabilities, let's look at an example of this in action. Suppose the large university has 24000 students and on a day late in a school year, 6000 of those students use the library.

Use your transition matrix A to determine how many of those 6000 will visit/not visit the library the next day.

Next, use A to determine how many of the 18000 who did not use the library will visit/not visit the library the next day.

Add together the numbers of students who will visit the library the next day. Do you get 6000?
Add together the numbers of students who will not visit the library the next day. Do you get 18000?

EXERCISE: Use the process from above(not your calculator) to find the Steady Probabilities the Transition Matrices: a) $\begin{bmatrix} 0.1 & 0.65 \\ 0.9 & 0.35 \end{bmatrix}$ b) $\begin{bmatrix} 1/4 & 2/5 \\ 3/4 & 3/5 \end{bmatrix}$

CALCULATOR EXERCISE: Use your calculator to raise the below Transition Matrix to each listed Power, and then use the →FRAC command to see if Fraction form is available yet

$$A = \begin{bmatrix} 0.3 & 0.6 & 0.4 \\ 0.7 & 0.3 & 0 \\ 0 & 0.1 & 0.6 \end{bmatrix}$$

i) A^{10} ii) A^{20} iii) A^{40} iv) A^{100}

EXERCISE: We revisit the gym member usage exercise from the Section 8.1 pages. Below is the Transition Matrix we built for this situation. Also given is the Distribution Matrix for Tuesday.

$$A = \begin{array}{c} \\ L \\ S \\ N \end{array} \begin{array}{c} \begin{array}{ccc} L & S & N \end{array} \\ \begin{bmatrix} .2 & .6 & .5 \\ .5 & .25 & .5 \\ .3 & .15 & 0 \end{bmatrix} \end{array} \qquad X_{Tuesday} = \begin{array}{c} L \\ S \\ N \end{array} \begin{bmatrix} .38 \\ .47 \\ .15 \end{bmatrix}$$

i) Show the necessary calculation for the following Tuesday, round-off values to the nearest 0.1%

ii) Find the %'s, rounded to the nearest 0.1, for a Tuesday 4 weeks later. And 8 weeks later.

iii) Now suppose the member usage was 5% long workout, 15% short, and 80% none on a day in winter when the city is hit by a massive blizzard. Build an appropriate Distribution Matrix, and find the %'s for 1 week, 4 weeks, and 8 weeks later. Compare to the results above in part (ii).

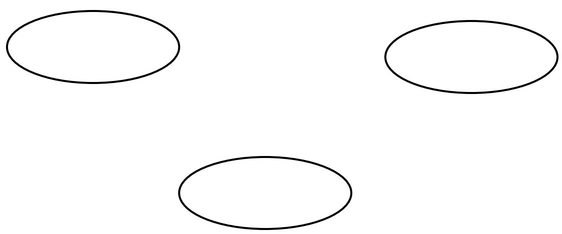
Once we get far enough forward from any random day, the values get pretty similar regardless of the Initial Distribution. So, rather than using an initial Distribution Matrix, X_0, which doesn't seem to matter anyway, just use A^{100} and convert the values to Fractions. You should have the same set of values in each column, which will match the values from any opening X_0 you may have chosen to use.

SECTION 8.3: ABSORBING STATES AND ABSORBING MATRICES

Earlier we looked at the concept of a Regular Stochastic Matrix, where it was considered Regular if some Power of the Stochastic Transition Matrix has ALL Non-Zero values. Raise each of the below Transition Matrices to the 20^{th}, 50^{th}, and 100^{th} powers.

$$A_1 = \begin{bmatrix} 0 & .2 & 0 \\ 0 & .3 & .4 \\ 1 & .5 & .6 \end{bmatrix} \qquad A_2 = \begin{array}{c} \\ E \\ F \\ G \end{array} \begin{array}{c} \begin{array}{ccc} E & F & G \end{array} \\ \begin{bmatrix} .5 & 0 & .4 \\ .2 & 1 & 0 \\ .3 & 0 & .6 \end{bmatrix} \end{array}$$

What do you see happening? A_1 is definitely Regular since at all three exponent levels, it has no Zeros, while A_2 never loses all of its Zeros. Therefore A_2 is not Regular. So, what is it? Build a Transition Diagram for Matrix A_2 using the bubbles below.



Do you see that state F never sends anything in the directions of either E or G? In fact, all it does is bring in from those states along with keeping hold of everything it already has. State F "Absorbs" anything that comes its way. Such states are accordingly called Absorbing States.

How do we identify Absorbing States? In the Transition Matrix, it is quite easy. Do you notice that State F has a 1 on the main diagonal of the matrix(from top left to bottom right)? Notice this means that 100% of what is currently in F this time will still be in F next time, nothing gets out, and has been "absorbed" by F. In your Diagram, F should have no arrows pointing towards anything but itself, again indicating that everything currently in F will again belong to F next time, all 100%.

So, what does this make matrix A_2? We would like to know if it is what we will now call an Absorbing Stochastic Matrix. It is Stochastic, but is it Absorbing? For this to be the proper description, it needs:
a) at least one Absorbing State. (Yes, we have already verified F is Absorbing)
b) ALL non-Absorbing states must have a pathway, directly or indirectly, to at least one Absorbing state.

Do E and G have pathways to F? The Diagram is perhaps easier to use in deciding this. Look at E in your Diagram. Notice it certainly has an arrow directly to F. But what about G? It does not have a direct pathway into F, no arrow in the Diagram from G to F. However, G does have an arrow to E, and as we noted, E has an arrow to F. This is an "indirect" pathway to F from G.

Conclusion: is Matrix A_2 an Absorbing Stochastic matrix?

EXERCISE: decide whether each of the following is Absorbing or not.

$$\text{i)}\quad \begin{array}{c} \\ U \\ V \\ W \\ Y \end{array} \begin{array}{c} \begin{array}{cccc} U & V & W & Y \end{array} \\ \begin{bmatrix} .2 & 0 & 0 & 0 \\ .1 & 0 & 1 & .5 \\ .3 & 1 & 0 & 0 \\ .4 & 0 & 0 & .5 \end{bmatrix} \end{array} \qquad \text{ii)}\quad \begin{array}{c} \\ J \\ K \\ L \\ M \end{array} \begin{array}{c} \begin{array}{cccc} J & K & L & M \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & .1 \\ .3 & 0 & 0 & .5 \\ .3 & 0 & 1 & 0 \\ .4 & 0 & 0 & .4 \end{bmatrix} \end{array}$$

Now we need to use an Absorbing Matrix to see what it tells us.

STEP 1: reorder the listings of the states across the top, and down the side, such that we list all of the absorbing states first, then the non-absorbing states. Call this form of the matrix the Standard Form. You may wish to use a Diagram to help the rewriting process.

$$\begin{array}{c} \\ J \\ K \\ L \\ M \end{array} \begin{array}{c} \begin{array}{cccc} J & K & L & M \end{array} \\ \begin{bmatrix} .3 & 0 & .2 & 0 \\ .3 & 1 & .4 & 0 \\ 0 & 0 & .3 & 0 \\ .4 & 0 & .1 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ K \\ M \\ J \\ L \end{array} \begin{array}{c} \begin{array}{cccc} K & M & J & L \end{array} \\ \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array} \qquad \left[\begin{array}{c|c} I & S \\ \hline 0 & R \end{array}\right]$$

Once you have reordered the matrix(one option of the list of states is given, others are possible), place lines in the matrix to help determine Matrices S and R, as shown in the Standard matrix setup above right.

STEP 2: Calculate the following: $S*(I-R)^{-1}$ using your calculator and place it back into the spot where S was in the Standard form matrix, and now we have the Stable matrix. It should look like this:

$$\left[\begin{array}{c|c} I & S*(I-R)^{-1} \\ \hline 0 & 0 \end{array}\right]$$

It's important to note which states are listed above $S*(I-R)^{-1}$, which are the Non-Absorbing states, as well as those along the side, which are the Absorbing states. The probabilities in $S*(I-R)^{-1}$ indicate the probabilities or proportions of values starting in each Non-Absorbing state that will eventually be absorbed into each of the Absorbing states.

Unlike when we looked at Regular matrices, now the amount starting in each of the Non-Absorbing states matters very much in determining what will eventually end up in the absorbing states.

STEP 3: Calculate $(I-R)^{-1}$, which is called the Fundamental matrix. Note that we still use the same Non-Absorbing state listing on top. Add the values in each column, where the total represents the expected number(or average) of transitions an element starting in that state will circulate until landing in one of the Absorbing states. It is not suggesting which Absorbing state.

EXERCISE: a 2-year college has been looking at the patterns of progress their students make towards completion, or not, of their degree. They find that for incoming freshman students, 30% do not take and/or pass enough credit hours to become sophomores, and so are freshmen the next year, 60% do pass enough and become sophomores, and 10% drop out and never return to school. Among those starting the year as sophomores, 50% earn enough credits to graduate, 35% do not earn enough to graduate, and so return the next year as sophomores again, and the other 15% drop out and never return.

STEP 1: Fill in the Transition Matrix. Then reorder it into Standard Form(use a Diagram if needed)

$$\begin{array}{c} \\ F \\ S \\ D \\ G \end{array} \begin{array}{c} \begin{array}{cccc} F & S & D & G \end{array} \\ \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array} \qquad\qquad \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

STEP 2: Identify Matrices S and R. Use your calculator to find $S*(I-R)^{-1}$, and place it into the proper upper right section of the now Stable Matrix below. Properly label the matrix.

$$\left[\begin{array}{c|c} I & \quad \\ \hline 0 & 0 \end{array}\right]$$

Suppose there are 5000 freshmen and 4000 sophomores at the start of this school year. Use the Stable Matrix probabilities to determine how many of the 9000 students total will eventually graduate and how many will eventually drop out(round to the nearest whole number, as needed).

$\#Graduate = \#F*\Pr(G) \ + \ \#S*\Pr(G) =$

$\#Dropout = \#F*\Pr(D) \ + \ \#S*\Pr(D) =$

STEP 3: Now calculate $(I-R)^{-1}$ using your calculator. Add up each column to determine the average number of years on average for Freshmen and Sophomores to spend at the school before eventually either graduating or dropping out.

EXERCISE: A consumer electronics company always has products in development. They have found that among products less than 1-year into development, 30% go to market, 60% get another year of funding, and 10% are abandoned. For projects over 1-year into development, they have found that 50% go to market, 10% get another year, and 40% abandoned.

a) Complete the labeling of the matrix below and fill in appropriate probabilities for this project cycle. Then convert the Matrix into Standard form.

$$\begin{array}{c} \begin{matrix} <1 & M & 1+ & Ab \end{matrix} \\ \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

b) Now identify matrices S and R. Use S and R to calculate $S*(I-R)^{-1}$, and place it into Stable Matrix form, properly labeled.

c) Suppose the company has 60 projects under 1-year in development and 90 projects over 1-year in development. Use your stable matrix to determine how many are expected to reach market and how many will eventually be abandoned.

d) Use R to calculate the Fundamental Matrix, $(I-R)^{-1}$. Use it to determine the expected number of years each of "less than 1-year" and "over 1-year" projects will spend in development before eventually either being brought to market or abandoned.