SECTION 2.1: SOLVING SYSTEMS OF EQUATIONS WITH A UNIQUE SOLUTION

In Chapter 1 we took a look at finding the intersection point of two lines on a graph. Chapter 2 begins with a look at a more formal approach to this same idea. The process is going to consist of using 3 operations. Using the system below, do each of the described operations, where each time you will still have 2 equations:

$$\begin{aligned} 2x+5y&=18 \\ x+3y&=10 \end{aligned}$$

1) Swap the 2 equations' positions, top-to-bottom and bottom-to-top.

2) Multiply both sides of the SECOND equation by 2.

3) The third operation involves adding a non-zero multiple of one equation to another equation. The purpose of this is to ***eliminate*** the variable in the "added to" equation. For example, you should have the system below after performing the first two operations, and we would now like to eliminate the *4x* term in the bottom equation. So, we add -4 times the top equation to the bottom one. NOTE: the top equation DOES NOT get changed, we just use it for our stated purpose.

$$\begin{aligned} x+3y&=10 \\ 4x+10y&=36 \end{aligned}$$

If you look at each of the operations, you will hopefully notice that it was just the Coefficients of *x* and *y*, as well as the right-side constants, that change. But the "columns" always represented *x*, *y*, and the constants. To simplify our work, we place just those coefficients and constants into a matrix. Going back to our original system:

$$\left.\begin{aligned} 2x+5y&=18 \\ x+3y&=10 \end{aligned}\right\} \qquad \left[\begin{array}{cc|c} 2 & 5 & 18 \\ 1 & 3 & 10 \end{array}\right]$$

Perform the same three operations as above on the values in the matrix, where some notation is given as a shorthand way to describe each operation:

1) $R_1 \leftrightarrow R_2$ 2) $2 \cdot R_2 \rightarrow R_2$ 3) $-4R_1 + R_2 \rightarrow R_2$

You should now have this partially reduced matrix: $\left[\begin{array}{cc|c} 1 & 3 & 10 \\ 0 & -2 & -4 \end{array}\right]$. The 1 in the top left corner is referred to as a "Leading 1" since it is the first non-Zero value in its Row. We would like to now move down a row from this leading 1, and then to the right and make that matrix element another leading 1. And once it is a leading 1, we will empty all other values in its column. Do so now, and write out notations to reflect each of the operations(Hint: it should take 2 op's) you perform.

Do you have this matrix? $\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right]$ Now, let us recall that the x-coefficients were placed in the first column and the y-coefficients into the second column. Those columns still, of course, represent those same variables. Turn each row of this final matrix back into its respective equation, and we have a unique solution to our original system of equations: x = 4 and y = 2

EXERCISE: Use the same process, in matrix form, including notations for each step, to solve:

$$3x + 4y = 25$$
$$x + 6y = 20$$

Did you get the solution (5 , 5/2) ??

PRACTICING ELEMENTARY ROW OPERATIONS:

$$\begin{bmatrix} 3 & -1 & 5 & -4 \\ 8 & 4 & -6 & 2 \\ -1 & 2 & -3 & 5 \end{bmatrix}$$ **Use this original Matrix to perform each Row Operation**

i) $R_2 \leftrightarrow R_3$

ii) $-4R_3 \to R_3$

iii) $3R_1 + R_2 \to R_2$

iv) $-2R_3 + R_1 \to R_1$

PIVOT: a "Pivot" is where at a chosen element in a matrix, we make that element a +1 and then proceed to use that 1 to eliminate all other values in its column(above or below). Perform a Pivot on the 2 in the 2nd Row and 4th column of the matrix in the previous exercise.

$$\begin{bmatrix} 3 & -1 & 5 & -4 \\ 8 & 4 & -6 & 2 \\ -1 & 2 & -3 & 5 \end{bmatrix}$$

EXERCISES: Use the process to reduce each system, turned into matrix form, and find the Unique solution to each.

a) $$\begin{aligned} 2x + y + 2z &= -1 \\ 3x + 2y + 3z &= -3 \\ x + y + 2z &= -3 \end{aligned}$$

b) $$\begin{aligned} 4x + y + z &= 3 \\ 2x + y + z &= 2 \\ 3x + y + 2z &= 1 \end{aligned}$$

SECTION 2.2: INFINITE SOLUTIONS AND NO SOLUTION

In Section 2.1 we saw that a System of Equations can have a Unique Solution of the form (x,y) or (x,y,z), and in fact can have even more variables and still have such a solution. Now, though, we will look at solving Systems of Equations which do not have such unique solutions.

Convert the system below into matrix form and follow the same process as we used in Section 2.1, where you should establish a "leading 1" in the top left corner, followed by another in Row 2, Column 2. Do so and then stop there.

$$\begin{aligned} 2x + y + 3z &= 8 \\ -x \qquad - z &= -3 \\ x + y + z &= 8 \end{aligned}$$

Do you get this: $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right]$?? What does the bottom Row suggest back in equation form?

We have a similar result to a situation we saw in Section 1.3, and the same conclusion of "No Solution"

Now try this System of Equations:

$$\begin{aligned} 3x + y + z - 7w &= 2 \\ x + y \qquad - 2w &= -3 \\ x + y + z + w &= -2 \\ 2x + y + z - 3w &= 4 \end{aligned}$$

Again, you find a Row suggesting a "false" equation, and we again conclude No Solution

Now try this System, where it is slightly different than the first example on the previous page, but the result is quite different.

$$\begin{aligned} 2x + y + 3z &= 8 \\ -x \quad - z &= -3 \\ x + y + z &= 5 \end{aligned}$$

You should get this: $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The Row of all 0's is actually not a problem, and can now be ignored. But take each of the first two Rows and convert them back into Equations with their proper variables(x, y, and z) back where they belong. $x+2z=3$ should be converted to $x=-2z+3$ and $y-z=2$ should be converted to $y=z+2$. Now, because we used *z* to describe both *x* and *y*, *z* can be "Any Real Number". This is an "Infinite Solution" because *z* has an infinite number of choices we can give it.

Now, reduce this system and find its properly represented Infinite Solution:
USE ROW OPERATIONS, NOT "RREF"

$$\begin{aligned} 3x + y + z - 7w &= 2 \\ x + y \quad - 2w &= -3 \\ x + y + z + w &= -2 \\ 2x + y + z - 3w &= 0 \end{aligned}$$

Convert the following System to a Matrix and reduce it to determine its Infinite Solution properly represented. "RREF" can be used

$$\begin{aligned} 2x + y + 5z &= -1 \\ x + y + 2z &= 1 \\ 3x + y + 8z &= -3 \end{aligned}$$

Did you get this: $\begin{cases} x = -3z - 2 \\ y = -z + 3 \\ z = \text{Any Real} \end{cases}$

Let's explore some possibilities.

Start with $z = 3$. Find the associated values for *x* and *y*. Check this solution in each of the 3 original equations to verify it is a valid solution.

Do the same where $z = -8$. Does the (x,y,z) ordered triple "check" in all 3 equations again?

Now try to find the particular solution if $y = -2$. Again, check it to verify it is indeed valid.

EXERCISES: Use "RREF" to solve each System of Equations. Show each reduced matrix and a PROPERLY STATED solution.

a) $\begin{aligned} 2x + 6y + z &= 10 \\ x + 3y &= 6 \\ 3x + 9y + z &= 16 \end{aligned}$

b) $\begin{aligned} 2x + 2y + z &= 9 \\ x + 2y + z &= 5 \\ 2x + 5y + 3z &= 9 \end{aligned}$

c) $\begin{aligned} 5x + 10y + 4z + 4w &= -7 \\ x + 2y + z + w &= -1 \\ -x - 2y - 2z - w &= -3 \\ 2x + 4y + 2z + 2w &= -2 \end{aligned}$

SECTION 2.3: OPERATIONS ON MATRICES

Do the following Matrix operations WITHOUT USING YOUR CALCULATORS

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} -3 & 5 \\ 4 & -2 \end{bmatrix}$$

i) $4A$

ii) $A + C$

iii) $3B + 2C$

iv) $2A - C + 4B$

EXERCISE: Find the values of Matrix E to make the equation true: $4F = 2E + D$

$$D = \begin{bmatrix} -2 & -8 \\ 6 & 4 \end{bmatrix} \qquad E = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad F = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}$$

MATRIX MULTIPLICATION

First, some practice on the Dot Product process. Multiply the matrices BY HAND

a) $$\begin{bmatrix} 2 & -1 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ -6 \end{bmatrix} = \big[(\quad)(\quad)+(\quad)(\quad)+(\quad)(\quad)\big] = \big[\qquad\big]$$

b) $$\begin{bmatrix} 5 & 10 & 7 \end{bmatrix}\begin{bmatrix} -6 \\ 5 \\ -8 \end{bmatrix} = \big[(\quad)(\quad)+(\quad)(\quad)+(\quad)(\quad)\big] = \big[\qquad\big]$$

Now, STILL BY HAND, multiply the following matrices, writing out each Dot Product in their respective slots in the extended matrix. Then find the simplified final product matrix.

$$\begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} 8 & 1 & 6 \\ 0 & 7 & 4 \end{bmatrix} = \left[\begin{array}{c:c:c} \qquad\qquad & \qquad\qquad & \qquad\qquad \\ \hdashline \qquad\qquad & \qquad\qquad & \qquad\qquad \end{array}\right] = \begin{bmatrix} \qquad\qquad\qquad\qquad \\ \\ \end{bmatrix}$$

SIZE/DIMENSIONS OF MATRICES and whether we can Add/Multiply matrices. Recall from lecture that we can only add matrices of exact same dimensions, the "sum matrix" being again these same dimensions. We can only multiply matrices where the column dimension of the first equals the row dimension of the second, with the "product matrix" having the row dimension of the first and column dimension of the second. We modeled the product in this fashion: $\underset{M\,x\,N}{A} \cdot \underset{N\,x\,P}{B} = \underset{M\,x\,P}{C}$. Given the below set of matrices, decide which calculations can or cannot be done according to proper dimension rules.

$$A = \begin{bmatrix} 1 & 9 \\ 8 & 2 \\ 3 & 7 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix},\ C = \begin{bmatrix} 7 \\ 4 \end{bmatrix},\ D = \begin{bmatrix} 2 & 8 \\ 6 & 4 \end{bmatrix},\ E = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 9 & 7 \\ 5 & 3 & 1 \end{bmatrix},\ F = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix},\ G = \begin{bmatrix} 5 & 4 & 3 \\ 2 & 1 & 0 \end{bmatrix}$$

For those that CAN be done, determine what are the dimensions of the final matrix.

i) $D+E+F$

ii) $AG-2E$

iii) $GE-AD$

iv) $CBE+G$

v) $DGAFE$

vi) $BA+CD$

EXERCISE: The matrix below lists 6 Math 125 students and the MyMathLab Percent averages they earned one semester on Homework, Quizzes, Midterms, and the Final Exam. The course webpage says homework is worth 80 points, quizzes 120, midterms 200, and the final exam 200, so a grand total of 600 points for the semester.

$$
\begin{array}{r c}
 & \begin{array}{cccc} HW & Qz & Mdtm & Fin \end{array} \\
\begin{array}{r} Amy \\ Bob \\ Chet \\ Dave \\ Eva \\ Fiona \end{array} &
\begin{bmatrix}
95.7 & 82.4 & 77.2 & 80 \\
56.8 & 84.8 & 73.6 & 75 \\
86.1 & 93.2 & 87.5 & 90 \\
76.3 & 94.1 & 72.8 & 85 \\
99.4 & 73.6 & 70.2 & 75 \\
96.8 & 92.4 & 78.2 & 70
\end{bmatrix}
\end{array}
$$

Construct any other necessary matrices(appropriately labeled) and give a Matrix expression(Hint: scalar multiplication might be needed) that will give each student their total points for the semester. How many points out of 600 maximum did each earn?