SECTION 2.1: SOLVING SYSTEMS OF EQUATIONS WITH A UNIQUE SOLUTION

In Chapter 1 we took a look at finding the intersection point of two lines on a graph. Chapter 2 begins with a look at a more formal approach to this same idea. The process is going to consist of using 3 operations. Using the system below, do each of the described operations, where each time you will still have 2 equations:

$$\begin{aligned} 2x+5y&=18 \\ x+3y&=10 \end{aligned}$$

1) Swap the 2 equations' positions, top-to-bottom and bottom-to-top.

2) Multiply both sides of the SECOND equation by 2.

3) The third operation involves adding a non-zero multiple of one equation to another equation. The purpose of this is to ***eliminate*** the variable in the "added to" equation. For example, you should have the system below after performing the first two operations, and we would now like to eliminate the *4x* term in the bottom equation. So, we add -4 times the top equation to the bottom one. NOTE: the top equation DOES NOT get changed, we just use it for our stated purpose.

$$\begin{aligned} x+3y&=10 \\ 4x+10y&=36 \end{aligned}$$

If you look at each of the operations, you will hopefully notice that it was just the Coefficients of x and y, as well as the right-side constants, that change. But the "columns" always represented x, y, and the constants. To simplify our work, we place just those coefficients and constants into a matrix. Going back to our original system:

$$\left.\begin{aligned} 2x+5y&=18 \\ x+3y&=10 \end{aligned}\right\} \qquad \left[\begin{array}{cc|c} 2 & 5 & 18 \\ 1 & 3 & 10 \end{array}\right]$$

Perform the same three operations as above on the values in the matrix, where some notation is given as a shorthand way to describe each operation:

1) $R_1 \leftrightarrow R_2$ 2) $2\cdot R_2 \to R_2$ 3) $-4R_1 + R_2 \to R_2$

You should now have this partially reduced matrix: $\left[\begin{array}{cc|c} 1 & 3 & 10 \\ 0 & -2 & -4 \end{array}\right]$. The 1 in the top left corner is referred to as a "Leading 1" since it is the first non-Zero value in its Row. We would like to now move down a row from this leading 1, and then to the right and make that matrix element another leading 1. And once it is a leading 1, we will empty all other values in its column. Do so now, and write out notations to reflect each of the operations(Hint: it should take 2 op's) you perform.

Do you have this matrix? $\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right]$ Now, let us recall that the x-coefficients were placed in the first column and the y-coefficients into the second column. Those columns still, of course, represent those same variables. Turn each row of this final matrix back into its respective equation, and we have a unique solution to our original system of equations: x = 4 and y = 2

EXERCISE: Use the same process, in matrix form, including notations for each step, to solve:

$$\begin{aligned} 3x + 4y &= 25 \\ x + 6y &= 20 \end{aligned}$$

Did you get the solution (5 , 5/2) ??

PRACTICING ELEMENTARY ROW OPERATIONS:

$$\begin{bmatrix} 3 & -1 & 5 & -4 \\ 8 & 4 & -6 & 2 \\ -1 & 2 & -3 & 5 \end{bmatrix}$$ **Use this original Matrix to perform each Row Operation**

i) $R_2 \leftrightarrow R_3$

ii) $-4R_3 \to R_3$

iii) $3R_1 + R_2 \to R_2$

iv) $-2R_3 + R_1 \to R_1$

PIVOT: a "Pivot" is where at a chosen element in a matrix, we make that element a +1 and then proceed to use that 1 to eliminate all other values in its column(above or below). Perform a Pivot on the 2 in the 2nd Row and 4th column of the matrix in the previous exercise.

$$\begin{bmatrix} 3 & -1 & 5 & -4 \\ 8 & 4 & -6 & 2 \\ -1 & 2 & -3 & 5 \end{bmatrix}$$

EXERCISES: Use the process to reduce each system, turned into matrix form, and find the Unique solution to each.

a)
$$\begin{aligned} 2x + y + 2z &= -1 \\ 3x + 2y + 3z &= -3 \\ x + y + 2z &= -3 \end{aligned}$$

b)
$$\begin{aligned} 4x + y + z &= 3 \\ 2x + y + z &= 2 \\ 3x + y + 2z &= 1 \end{aligned}$$

SECTION 2.2: INFINITE SOLUTIONS AND NO SOLUTION

In Section 2.1 we saw that a System of Equations can have a Unique Solution of the form (x,y) or (x,y,z), and in fact can have even more variables and still have such a solution. Now, though, we will look at solving Systems of Equations which do not have such unique solutions.

Convert the system below into matrix form and follow the same process as we used in Section 2.1, where you should establish a "leading 1" in the top left corner, followed by another in Row 2, Column 2. Do so and then stop there.

$$\begin{aligned} 2x + y + 3z &= 8 \\ -x \quad - z &= -3 \\ x + y + z &= 8 \end{aligned}$$

Do you get this: $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right]$?? What does the bottom Row suggest back in equation form?

We have a similar result to a situation we saw in Section 1.3, and the same conclusion of "No Solution"

Now try this System of Equations:

$$\begin{aligned} 3x + y + z - 7w &= 2 \\ x + y \quad - 2w &= -3 \\ x + y + z + w &= -2 \\ 2x + y + z - 3w &= 4 \end{aligned}$$

Again, you find a Row suggesting a "false" equation, and we again conclude No Solution

Now try this System, where it is slightly different than the first example on the previous page, but the result is quite different.

$$\begin{aligned} 2x + y + 3z &= 8 \\ -x \qquad - z &= -3 \\ x + y + z &= 5 \end{aligned}$$

You should get this: $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The Row of all 0's is actually not a problem, and can now be ignored. But take each of the first two Rows and convert them back into Equations with their proper variables(x, y, and z) back where they belong. $x + 2z = 3$ should be converted to $x = -2z + 3$ and $y - z = 2$ should be converted to $y = z + 2$. Now, because we used *z* to describe both *x* and *y*, *z* can be "Any Real Number". This is an "Infinite Solution" because *z* has an infinite number of choices we can give it.

Now, reduce this system and find its properly represented Infinite Solution:
USE ROW OPERATIONS, NOT "RREF"

$$\begin{aligned} 3x + y + z - 7w &= 2 \\ x + y \qquad - 2w &= -3 \\ x + y + z + w &= -2 \\ 2x + y + z - 3w &= 0 \end{aligned}$$

Convert the following System to a Matrix and reduce it to determine its Infinite Solution properly represented. "RREF" can be used

$$\begin{aligned} 2x+y+5z&=-1 \\ x+y+2z&=1 \\ 3x+y+8z&=-3 \end{aligned}$$

Did you get this: $\begin{cases} x=-3z-2 \\ y=-z+3 \\ z=\text{Any Real} \end{cases}$

Let's explore some possibilities.

Start with $z=3$. Find the associated values for x and y. Check this solution in each of the 3 original equations to verify it is a valid solution.

Do the same where $z=-8$. Does the (x,y,z) ordered triple "check" in all 3 equations again?

Now try to find the particular solution if $y=-2$. Again, check it to verify it is indeed valid.

EXERCISES: Use "RREF" to solve each System of Equations. Show each reduced matrix and a PROPERLY STATED solution.

a) $\begin{aligned} 2x+6y+z&=10 \\ x+3y\quad\;&=6 \\ 3x+9y+z&=16 \end{aligned}$

b) $\begin{aligned} 2x+2y+z&=9 \\ x+2y+z&=5 \\ 2x+5y+3z&=9 \end{aligned}$

c) $\begin{aligned} 5x+10y+4z+4w&=-7 \\ x+2y+z+w&=-1 \\ -x-2y-2z-w&=-3 \\ 2x+4y+2z+2w&=-2 \end{aligned}$

SECTION 2.3: OPERATIONS ON MATRICES

Do the following Matrix operations WITHOUT USING YOUR CALCULATORS

$$A=\begin{bmatrix}2 & -3\\ -1 & 6\end{bmatrix} \qquad B=\begin{bmatrix}4 & 1\\ -5 & 2\end{bmatrix} \qquad C=\begin{bmatrix}-3 & 5\\ 4 & -2\end{bmatrix}$$

i) $4A$

ii) $A+C$

iii) $3B+2C$

iv) $2A-C+4B$

EXERCISE: Find the values of Matrix E to make the equation true: $4F=2E+D$

$$D=\begin{bmatrix}-2 & -8\\ 6 & 4\end{bmatrix} \qquad E=\begin{bmatrix}a & b\\ c & d\end{bmatrix} \qquad F=\begin{bmatrix}3 & 1\\ 2 & -2\end{bmatrix}$$

MATRIX MULTIPLICATION

First, some practice on the Dot Product process. Multiply the matrices BY HAND

a) $$\begin{bmatrix} 2 & -1 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ -6 \end{bmatrix} = \left[(\quad)(\quad)+(\quad)(\quad)+(\quad)(\quad)\right] = \left[\qquad\right]$$

b) $$\begin{bmatrix} 5 & 10 & 7 \end{bmatrix}\begin{bmatrix} -6 \\ 5 \\ -8 \end{bmatrix} = \left[(\quad)(\quad)+(\quad)(\quad)+(\quad)(\quad)\right] = \left[\qquad\right]$$

Now, STILL BY HAND, multiply the following matrices, writing out each Dot Product in their respective slots in the extended matrix. Then find the simplified final product matrix.

$$\begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} 8 & 1 & 6 \\ 0 & 7 & 4 \end{bmatrix} = \left[\begin{array}{c:c:c} \qquad & \qquad & \qquad \\ \hdashline \qquad & \qquad & \qquad \end{array}\right] = \left[\qquad\qquad\qquad\right]$$

SIZE/DIMENSIONS OF MATRICES and whether we can Add/Multiply matrices. Recall from lecture that we can only add matrices of exact same dimensions, the "sum matrix" being again these same dimensions. We can only multiply matrices where the column dimension of the first equals the row dimension of the second, with the "product matrix" having the row dimension of the first and column dimension of the second. We modeled the product in this fashion: $\underset{M\,x\,N}{A} \cdot \underset{N\,x\,P}{B} = \underset{M\,x\,P}{C}$. Given the below set of matrices, decide which calculations can or cannot be done according to proper dimension rules.

$$A = \begin{bmatrix} 1 & 9 \\ 8 & 2 \\ 3 & 7 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix},\ C = \begin{bmatrix} 7 \\ 4 \end{bmatrix},\ D = \begin{bmatrix} 2 & 8 \\ 6 & 4 \end{bmatrix},\ E = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 9 & 7 \\ 5 & 3 & 1 \end{bmatrix},\ F = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix},\ G = \begin{bmatrix} 5 & 4 & 3 \\ 2 & 1 & 0 \end{bmatrix}$$

For those that CAN be done, determine what are the dimensions of the final matrix.

i) $D+E+F$

ii) $AG-2E$

iii) $GE-AD$

iv) $CBE+G$

v) $DGAFE$

vi) $BA+CD$

EXERCISE: The matrix below lists 6 Math 125 students and the MyMathLab Percent averages they earned one semester on Homework, Quizzes, Midterms, and the Final Exam. The course webpage says homework is worth 80 points, quizzes 120, midterms 200, and the final exam 200, so a grand total of 600 points for the semester.

$$\begin{array}{r c} & \begin{array}{cccc} HW & Qz & Mdtm & Fin \end{array} \\ \begin{array}{r} Amy \\ Bob \\ Chet \\ Dave \\ Eva \\ Fiona \end{array} & \begin{bmatrix} 95.7 & 82.4 & 77.2 & 80 \\ 56.8 & 84.8 & 73.6 & 75 \\ 86.1 & 93.2 & 87.5 & 90 \\ 76.3 & 94.1 & 72.8 & 85 \\ 99.4 & 73.6 & 70.2 & 75 \\ 96.8 & 92.4 & 78.2 & 70 \end{bmatrix} \end{array}$$

Construct any other necessary matrices(appropriately labeled) and give a Matrix expression(Hint: scalar multiplication might be needed) that will give each student their total points for the semester. How many points out of 600 maximum did each earn?

SECTION 2.4: THE INVERSE OF A MATRIX PLUS APPLICATION

It was described in class that two Square Matrices, A and B, are Inverses of one another if $AB = BA = I$, where I is the appropriately sized Identity Matrix.

First, as a demonstration of the property, verify that the two matrices below are in fact Inverses of one another, by multiplying them together in both orders, $AB \text{ and } BA$. FOR GOOD PRACTICE, DO THIS BY HAND, WRITING OUT EACH DOT PRODUCT FULLY.

$$A = \begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix} \quad , \quad B = \begin{bmatrix} -4 & 3 \\ 7 & -5 \end{bmatrix}$$

EXERCISE: Decide whether each pair below are/are not Inverses, still using the concept: $AB = BA = I$
To speed up your work, use calculators this time.

i) $A = \begin{bmatrix} 4 & 3 \\ 6 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 5/2 & -3/2 \\ -3 & 2 \end{bmatrix}$

ii) $A = \begin{bmatrix} 16 & 6 \\ 10 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -5/2 \\ -3/2 & 4 \end{bmatrix}$

iii) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 3 & -1 \end{bmatrix}$

FINDING THE INVERSE OF A 2x2 MATRIX BY USE OF A FORMULA

In class, the below formula was given for finding the Inverse of a 2x2 Matrix. Use it to find the Inverse of each matrix. **It is highly recommended you determine *ad-bc* first**.

$$if \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad , then \;\; A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \; , \; ad-bc \neq 0$$

i) $\begin{bmatrix} 4 & 3 \\ 6 & 5 \end{bmatrix}$

ii) $\begin{bmatrix} 16 & 6 \\ 10 & 4 \end{bmatrix}$

iii) $\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$

iv) $\begin{bmatrix} -2 & 4 \\ 0 & M \end{bmatrix} \; M \neq 0$

v) $\begin{bmatrix} -2 & 3/2 \\ 3/2 & -1 \end{bmatrix}$

AN APPLICATION FOR USING THE INVERSE

Earlier in the chapter, we explored solving Systems of Linear Equations and solved them by use of Gaussian Elimination. We have now shown in class that if there is a Unique Solution, it can be found by converting the System of Equations into Matrices *A, X, and B*. The matrix Equation $AX = B$ represents the actual system, and it was shown that *X*, our desired solution, could be found by $X = A^{-1}B$

Given the system $\begin{matrix} 7x+3y=23 \\ 9x+4y=30 \end{matrix}$, deconstruct the system into matrices *A, X, and B.*

$A = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \qquad X = \begin{bmatrix} \quad \\ \quad \end{bmatrix} \qquad B = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$

Now, find the Inverse to A, by hand using the formula:

Next, determine the values in X by use of $X = A^{-1}B$, performing the calculations BY HAND.

Finally, check your solution back in the original equations to verify it is the correct Unique Solution.

EXERCISE: Redo the above process on this system: $\begin{matrix} 4x+3y=13 \\ 8x+5y=25 \end{matrix}$

SECTION 2.5: GAUSS-JORDAN METHOD OF FINDING THE INVERSE

The Gauss-Jordan method for finding an Inverse to a Matrix is quite similar to the Gaussian Elimination process we used for solving a System of Equations earlier in the chapter. All our decision in Gaussian Elimination were made based on the values to the left of our “Augment Bar”, and exactly the same thing will occur in Gauss-Jordan. Use Gauss-Jordan process to find the Inverse of A, the left side of the matrix already augmented below with an appropriate sized Identity Matrix.

Write out proper notations of each Elementary Row Operation and show all intermediate matrices. Once finished, verify by use of your calculators. You will not necessarily need all of the provided matrices.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Now that we have found the Inverse to $\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \\ 2 & -1 & 3 \end{bmatrix}$, specifically finding $\begin{bmatrix} 5 & -7 & -2 \\ 4 & -5 & -2 \\ -2 & 3 & 1 \end{bmatrix}$, use your calculators to verify that multiplying them in either order does give you a 3x3 Identity matrix.

Next, suppose you are asked to solve the below Systems of Equations WITHOUT USING YOUR CALCULATORS. Apply the concept from Section 2.4: $X = A^{-1}B$ by appropriately deciding which of the matrices is A and which is A^{-1}

i) $$\begin{aligned} x + y + 4z &= 18 \\ y + 2z &= 10 \\ 2x - y + 3z &= 9 \end{aligned}$$

ii) $$\begin{aligned} 5x - 7y - 2z &= 38 \\ 4x - 5y - 2z &= 28 \\ -2x + 3y + z &= -15 \end{aligned}$$

EXERCISE: Use Gauss-Jordan to find the inverse of $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$

SECTION 2.6: LEONTIEFF INPUT-OUTPUT ANALYSIS

Discussed in lecture was the idea of wanting to determine how much total production from various industries, or divisions of a company, would be necessary to not only satisfy the internally used output, but also have enough to satisfy the needs of the public as well.

The Input-Output values were placed into a square matrix, A, where we must label the matrix both across the top and down the side in exactly the same order. The industry listings on top refer to the one "Producing $1 of Output", while the listing on the side of the matrix refer to the industry where "Input is needed from".

For the problem presented, first construct and fully label a matrix, and then fill in the appropriate values. WARNING: you should never assume information is given "in order".

A small country has two industries, Bacon and Eggs. To produce $1 of bacon, that industry needs $0.10 of their own output and $0.20 from the eggs industry, while to produce $1 of eggs, that industry needs $0.10 of their own output and $0.30 from the bacon industry.

$A = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$ and now $I - A = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$

Next, use the formula from back in Section 2.4 to find the Inverse: $(I-A)^{-1}$ in FRACTIONS

Now suppose we are told that the population needs $36000 in bacon and $30000 in eggs. Build an appropriately labeled matrix D for these values. Then determine total production, $X = (I-A)^{-1} D$

What is the economy has more than 2 industries? The Inverse Matrix needed is far too complicated for us to determine by the methods we have studied, the formula for a 2x2 Inverse won't work, and the gauss-Jordan process would force us to work with incredibly large numbers in the numerator and denominators of fractions. So, this example will mirror what MML demands of you in determining total production values.

An economy consists of three industries: metals, plastic, and energy. To produce $1 of metal, that industry needs $0.12 of their own output, $0.02 from plastic, and $0.25 from energy. To produce $1 of output, the plastics industry needs $0.08 of their own output, $0.15 from metals, and $0.18 from energy. To produce $1 of output, the energy industry needs $0.05 of their own output and $0.09 from the metals industry. The public needs each year $65 million in metals, $48 million in plastic, and $82 million in energy. How much must each industry produce in total to satisfy this demand?

a) Build a fully labeled Input-Output matrix, *A*. Make sure you line up properly which industry is producing $1 of output to which industry is providing a specified amount of input.

b) Use your calculator to calculate $(I-A)^{-1}$. The command should look like this: $(identity(3)-[A])^{-1}$ Now you need to round each of the decimal values in $(I-A)^{-1}$ to 2-decimal places. The best way to do this is save the result into Matrix [B] using the "store" key on the calculator. Then go into "Edit" on Matrix *B* and round each value you see down to 2-decimal places. Write the "rounded" matrix below.

c) Build and label matrix *D*, the public demand matrix. USE the given values, "in millions", NOT full length values with all of the 0's. So, for example, 65, **NOT 65000000**. Now multiply the rounded-off matrix with *D*. Do you get the below values in *X*?

$$\begin{bmatrix} 95.15 \\ 54.27 \\ 121.19 \end{bmatrix}$$

If you did not get this, check that *A* was properly built as well as your rounding in $(I-A)^{-1}$.

Question: Would we ever want to look this problem from the industry end of the situation? Meaning, what if we knew how much total production, X, is possible, and want to then see what is actually available for public consumption? Recall that we started the whole process by assuming total production minus a calculation of "internal industry usage" would leave an amount, D, for the public. In that discussion, we created the equation $X - AX = D$, which was then converted into $X = (I - A)^{-1} D$, the calculation we have used in the above examples. Let us look at an example using $X - AX = D$.

Olympia Island has 3 industries, Gold, Silver, and Bronze. To produce \$1 of Gold, that industry needs \$0.03 of gold, \$0.06 of silver, and \$0.10 of bronze. To produce \$1 of Silver, that industry needs \$0.07 of gold, \$0.09 of silver, and no bronze. To produce \$1 of Bronze, that industry needs \$0.02 of gold, \$0.12 of silver, and \$0.08 of bronze. Build A, the Input-Output matrix.

a) Suppose the three industries have the capacities to produce only \$60000 in gold, \$50000 in silver, and \$75000 in bronze. Use $X - AX = D$ to determine how much of this total output would be left over for the public.

b) How much of that total production is then used internally by the three industries?

c) Now suppose we do know the public demand figures of \$90000 in gold, \$82000 in silver, and \$68000 in bronze. Use A as created above and the methods in the example on the previous page, including the 2-decimal rounding of $(I - A)^{-1}$, to determine what total production, X, is necessary.