SECTION 1.1: Plotting Coordinate Points on the X-Y Graph

This should be a review subject, as it was covered in the prerequisite coursework. But as a reminder, and for practice, plot each of the following Coordinate Pairs in (x, y) form by first moving in direction *x* and then in direction *y* the distance specified by each *x* or *y* value. The axes are not marked, so first place "tick-marks" along all 4 directions, evenly spaced, and then number them by 1's(1, 2,3… or -1, -2, -3…)

$(2,3)$

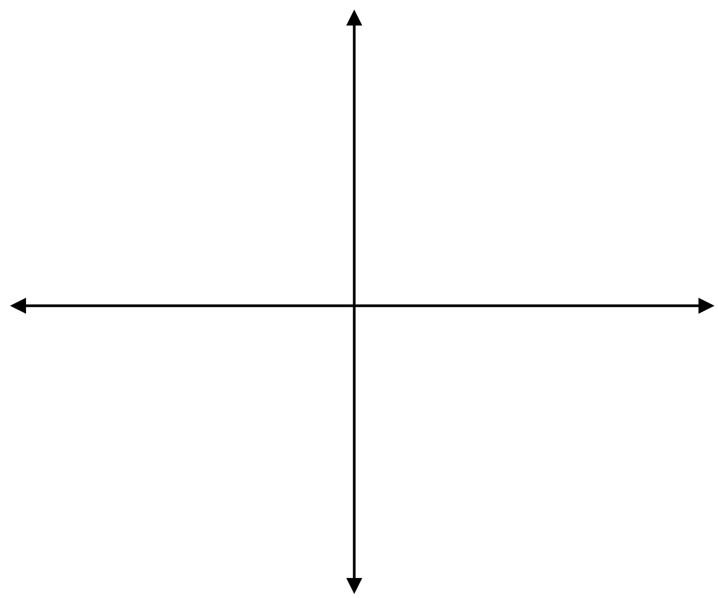
$(-1,2)$

$(0,1)$

$(4,0)$

$(-2,-3)$

$(3,-2)$



Now let's try some Coordinate Pairs that resemble ones you will definitely see later in this course, ones that stretch further out than 2 or 3 in either direction. Before jumping the gun, stop and think how you will mark the axes. By 1's like the above exercise? Not much use, right? So, make each tick-mark 10 apart from the next. You will have to place some values between successive marks, but that's OK.

$(25,0)$

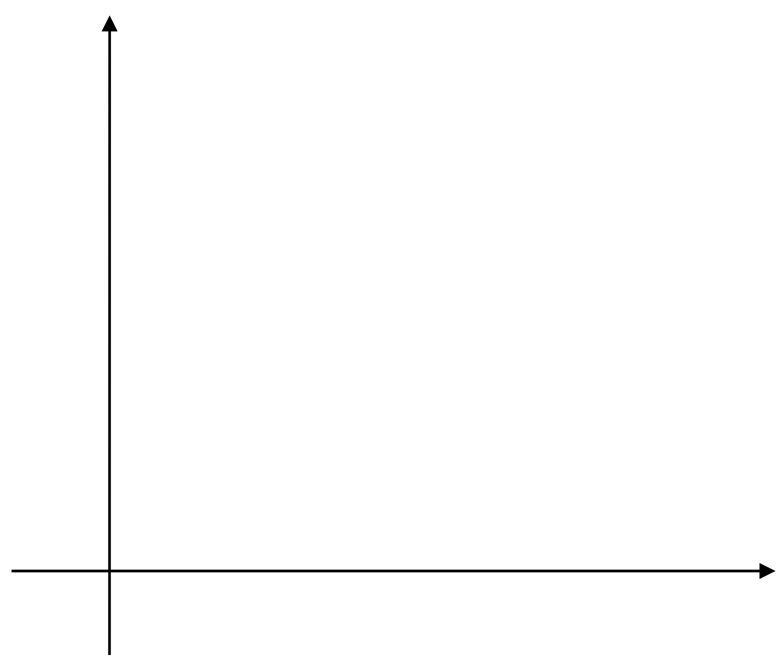
$(40,0)$

$(56,0)$

$(0,32)$

$(0,20)$

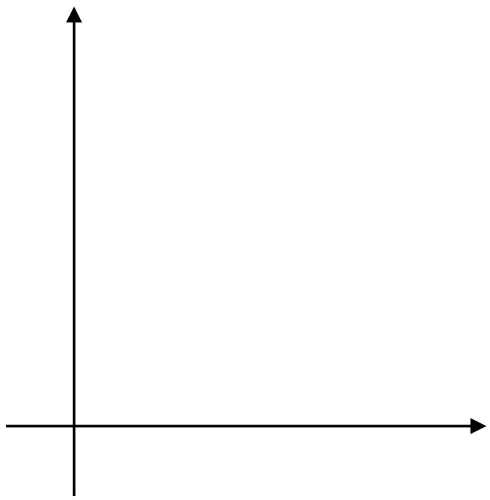
$(0,60)$



Note that ALL of these points are Intercepts(along an axis), something you should see often in later work.

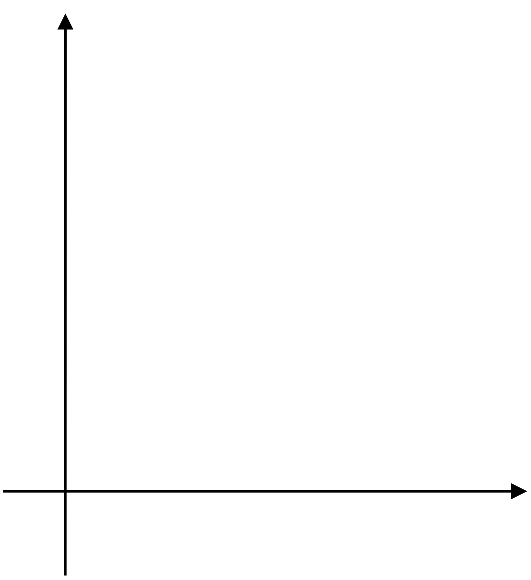
Now let's look at graphing lines. There are many forms the equation of a line, like "General", which has the look $cx + dy = e$ and Slope-Intercept, with the look $y = mx + b$. While the book and MyMathLab will tell you to turn the General form into Slope-Intercept form, I would prefer you use the following method, since it works best for the types of equations you will encounter most often later in this course.

Start with the equation $6x + 2y = 12$. There is really no need to convert to Slope-Intercept, when finding both Intercepts is relatively easy. It gives us two useful points to plot, plenty to sketch a decent graph. Start by plugging in a 0 for *y*, and solve for *x*, put in $(x, 0)$ form. Then plug in a 0 for *x*, and solve for *y*, and put in $(0, y)$. Appropriately mark both axes, with numbers, and list each coordinate pair by its plotted point. Then sketch a line through both.

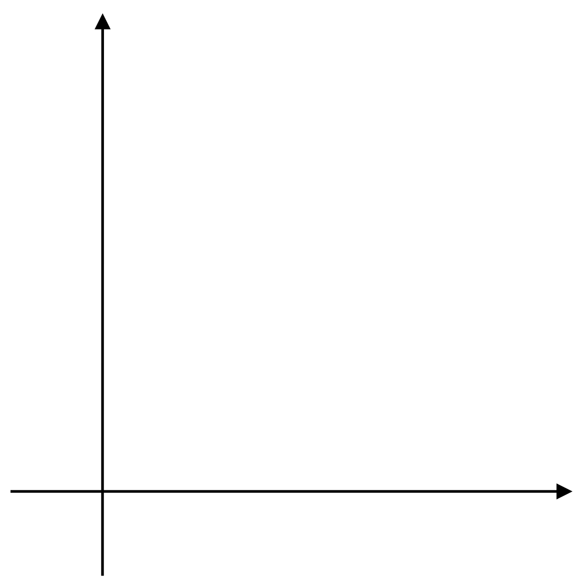


Now, do the same thing for each equation below. How will you mark/number each axis? By 1's? Hopefully not, but don't decide until you have the Intercepts. Should they be done the same? (No)

$12x + 6y = 168$



$10x + 15y = 900$



1.1 EXERCISES

1) Determine whether each coordinate pair is on the line $4x+7y=224$

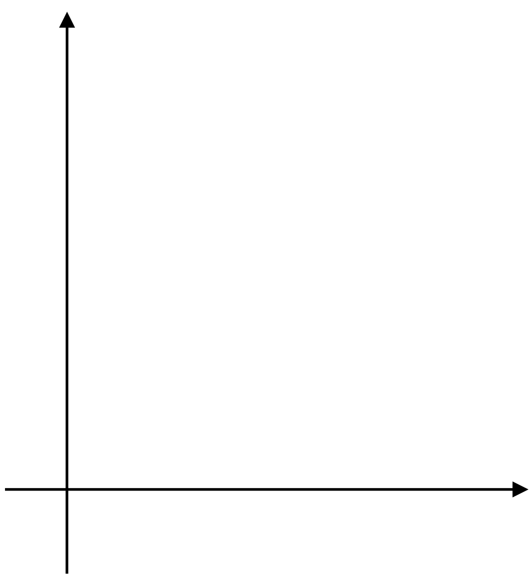
a) $(54,0)$ b) $(50,2)$ c) $(32,16)$ d) $(77/2,10)$ e) $(12,20)$ f) $(16,23)$

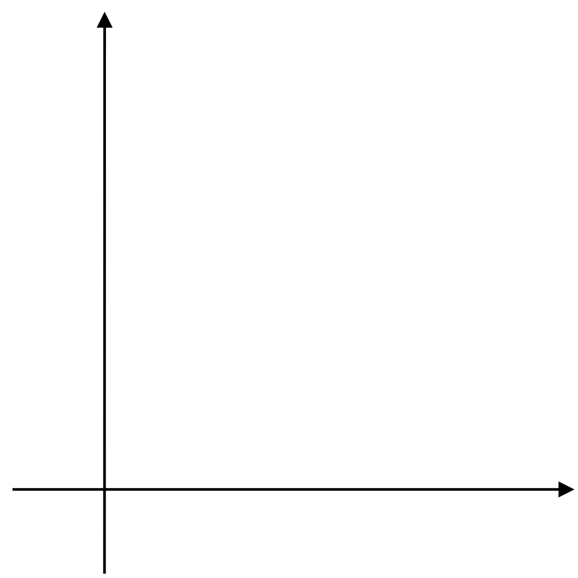
2) Find BOTH Intercepts, in Coordinate form, for each Linear Equation: Fractions only, NO Decimals

a) $24x+18y=864$ b) $20x+45y=1200$ c) $6x+10y=75$

3) Graph each Linear Equation by finding the Intercepts. Carefully mark and number your axes!!

a) $2x+6y=180$ b) $10x+5y=210$





4) Put each line into Slope-Intercept form

a) $8x+6y=96$ b) $4x+18y=90$

SECTION 1.2: SLOPE OF A LINE AND CREATING LINEAR EQUATIONS

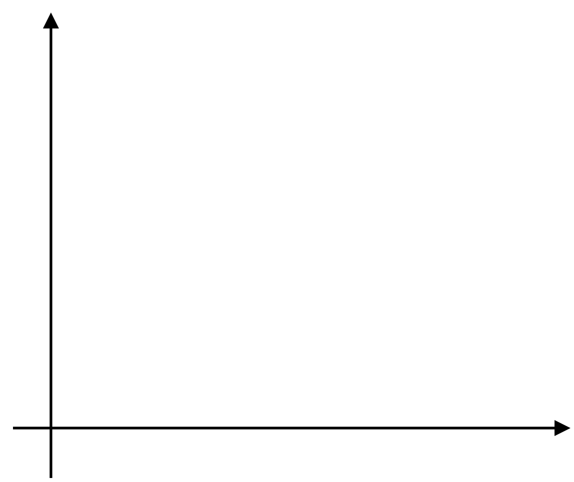
Most of the Equations you will use in this course will be in the General Form, $cx+dy=e$. The Slope of the Line is not visible in this form, but can be quite easily be found by converting the equation into Slope-Intercept form, $y=mx+b$, in a simple 2-step process: (i) "move" the term with *x* to the other side, and (ii) divide both sides of the equation by *d*. Do so on the following examples:

a) $4x+3y=24$ b) $2x+6y=60$ c) $8x+20y=600$

In each of the examples, you should now have an equation in $y=mx+b$ form, where *m*, the coefficient of *x*, is your Slope: (a) -4/3, (b) -1/3, (c) -2/5

The Slope is a very important value of any Line, offering us valuable information in certain contexts and usages of the Line. But, is it useful for graphing the typical Line we will see in this course? Let us graph the line in letter (c) above two ways.

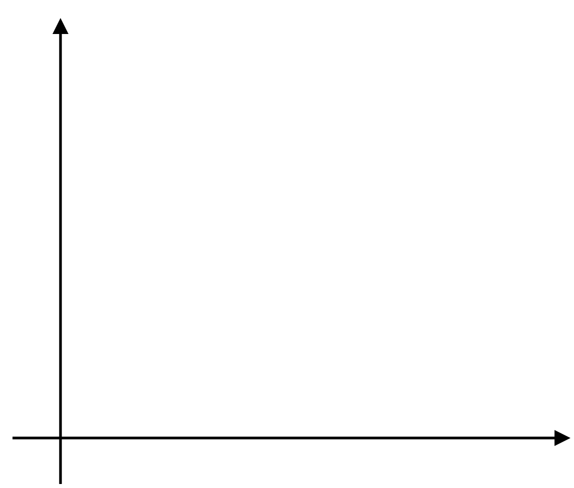
i) Using Slope. You should have the equation $y=\frac{-2}{5}x+30$. So, the 30 indicates a y-intercept at (0, 30). Plot this and then use the slope to go down 2 and right 5 to plot a second point. Draw the line through these 2 points.



ii) Starting with $8x+20y=600$, find both intercepts like we did in the 1.1 worksheet. Plug in 0 for *y* to get the *x-intercept* and then plug in 0 for *x* to get the *y-intercept*.

How will you number your axes? Try using 10's

Does the line in (i) hit the X-Axis at the same point as your *x-intercept* in (ii)? Did it even hit the X-Axis?



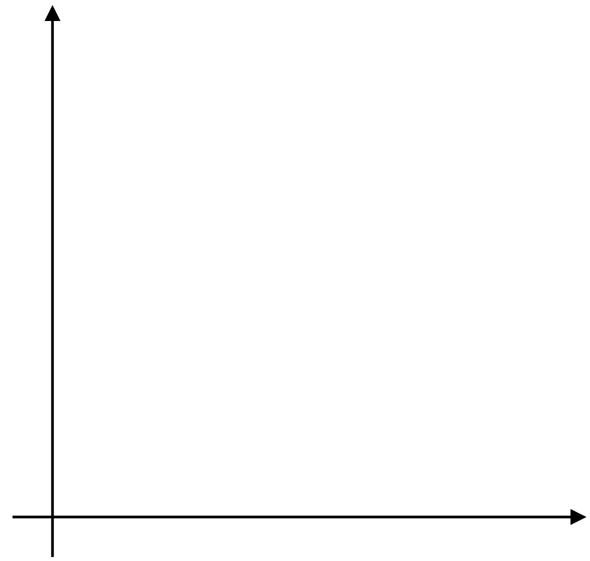
The previous example is meant to show that when we can find BOTH intercepts, it is a superior choice to graphing the typical line we will encounter in this course. However, the lines we have been graphing in this way, in $cx + dy = e$ form have all had $e \neq 0$. What about when $e = 0$? There will be only one Intercept, which is at (0, 0). So, how should we approach graphing these lines? Again, slope can be used but might not be the best choice when "larger" values are in play. Let us graph $2x - 3y = 0$, but suppose we also are graphing the Line $4x + 5y = 600$.

The Line $4x + 5y = 600$ has as its Intercepts $(150,0)$ *and* $(0,120)$, which you should verify for yourself. How should we number our Axes for these Intercepts? By 1's? By 10's? Maybe by 30's? Use 30's.

Draw the line through $(150,0)$ *and* $(0,120)$

Now take $2x - 3y = 0$, what is the Slope? Do you get $m = 2/3$? Try using the Slope and (0, 0), moving Up 2 and Right 3. Does this work very well? Probably not.

So, what if we decide to use $x = 60$? What is y? You should get y = 40, so the point (60,40). Plot (60, 40) and decide whether your line using "Up 2 and Right 3" was accurate or not.



Try this on the following pairs of lines, numbering the Axes as suggested in each.

$2x - y = 0$

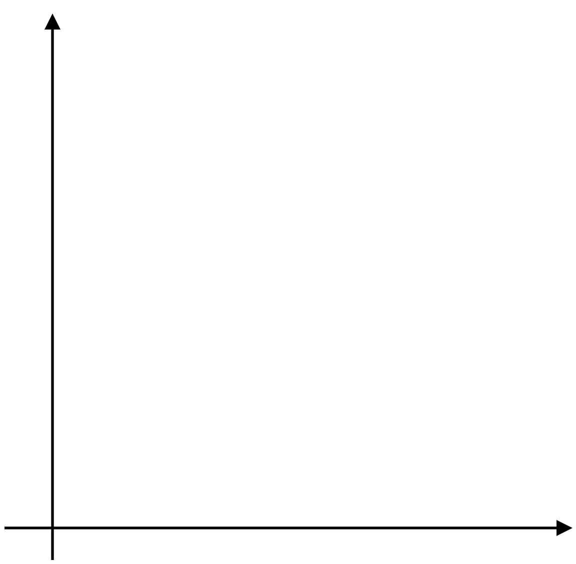
$5x + 10y = 500$

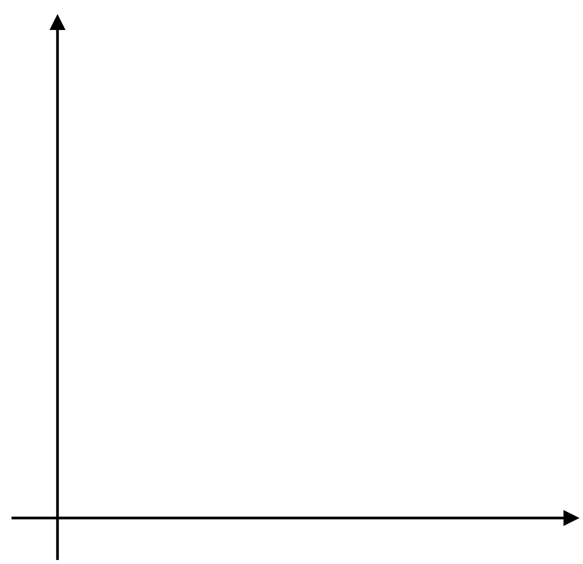
By 25's

$x - 3y = 0$

$12x + 18y = 540$

By 10's





The Slope of a Line is relatively simple to determine when we already have an equation for the Line, but what about when we do not? For this situation, we go back to an Algebra formula:

$Slope = m = \frac{y_2 - y_1}{x_2 - x_1}$, where the *x's* and *y's* are from two separate coordinate pairs.

Practice use of the formula on each of the below sets of coordinates(leave answers in FRACTIONS):

a) $(4,10)$ *and* $(8,20)$ b) $(0,16)$ *and* $(8,12)$ c) $(30,125)$ *and* $(45,210)$

Now, let us build an equation for a Line when all we have to start are two coordinate pairs. Again, we go back to an Algebra formula: $y - y_1 = m(x - x_1)$. Recall that *m* is our slope, determined by the formula above, and that the x_1 *and* y_1 come from EITHER coordinate pair. Try this on each of the previous examples, using the Slopes you determined above.

a) $(4,10)$ *and* $(8,20)$ b) $(0,16)$ *and* $(8,12)$ c) $(30,125)$ *and* $(45,210)$

APPLICATION: Suppose a company determines that it costs $3200 to build 40 items, and it costs $4950 to build 75 items. They would like to have a Linear Cost Equation in Slope-Intercept form. What should be *x* and what should be *y*? Ask yourself this: "Is a cost of $3200 the result of building 40 items, or is building 40 items the result of a cost of $3200?" The first makes more sense, doesn't it? Use that idea to build the two coordinate pairs, find the Slope, and lastly build the full equation in $y = mx + b$ form.

Note: the value of *m* is now a "per item cost", while *b* is reflective of a "fixed cost".

SECTION:1.3 THE INTERSECTION POINT OF TWO LINES

The Coordinates of a point of Intersection of two lines is often of great importance to us, so we want to use proper methods to determine these Coordinates, as opposed to "guessing" by just looking at a graph. The textbook and MyMathLab "help" will suggest one method, which we will use below, but then compare it to the method you should have learned in Algebra.

Find the Intersection of the two Lines: $\begin{array}{l} 2x+5y=16 \\ 7x+3y=27 \end{array}$

Method I(text/MML): solve each Equation for *y*, into Slope-Intercept form. Once you have done so, use the Transitive Property to set the "*x* sides" equal to one another. Solve this equation for *x*. Once you have *x*, plus it back into an original equation to solve for *y*.

Did you get $\frac{-2}{5}x+\frac{16}{5}=\frac{-7}{3}x+9$? How fun was it to solve this fraction-filled equation?

Method II(from Algebra): Choose either Variable, *x* or *y*, and multiply each equation, on both sides, by the Coefficient from the other equation, introducing a Minus Sign to one of them, if necessary, so that you will now have +/- the same coefficient on that chosen variable. Let us select *x*, which means we multiply the first equation by 7, and the second by -2(to address the need for a Minus Sign). You should have a +14*x* and a -14*x* in your rewritten equations. Add those equations together, eliminating *x*. Then solve for *y*, and then use y plugged back into an original equation to determine *x*.

Were there any fractions to be handled in this process? Which of the 2 methods do you prefer?

Try each method again on the following exercises. Either will be allowed on Quizzes and Exams, this worksheet is simply trying to help you decide which works best for yourself.

a) $\begin{array}{l} 4x+5y=13 \\ x+3y=5 \end{array}$ b) $\begin{array}{l} 2x-3y=6 \\ 5x+y=32 \end{array}$ c) $\begin{array}{l} 12x+15y=195 \\ 20x+10y=250 \end{array}$ d) $\begin{array}{l} 8x+9y=26 \\ 12x+3y=32 \end{array}$

Application: Suppose two salespeople are newly hired by a company. The first asks for a weekly base salary of $225 plus a commission of $35 for each sale made, while the second one asks for a weekly base of $150 plus a commission of $40 per sale. (i) Build linear Equations to represent each salesperson's weekly earnings for sales of *x* items, and (ii) Find the number of sales that would see them with identical earnings for a week, plus the total weekly earnings for each in that case.

Application: A family goes to a ballgame and buys 4 sandwiches and 5 sodas for a total of $43.30, while another family purchased 6 sandwiches and 4 sodas for a total of $52.70. Build two equations using X = Price of a Sandwich and Y = Price of a Soda, and then solve for these two prices.

1.3 EXERCISES

1) Use Elimination to find the intersection of each pair of lines

a) $\begin{aligned} x+5y&=14 \\ 7x+3y&=34 \end{aligned}$

b) $\begin{aligned} 6x+5y&=35 \\ 4x+3y&=22 \end{aligned}$

c) $\begin{aligned} 10x+15y&=170 \\ 25x+20y&=320 \end{aligned}$

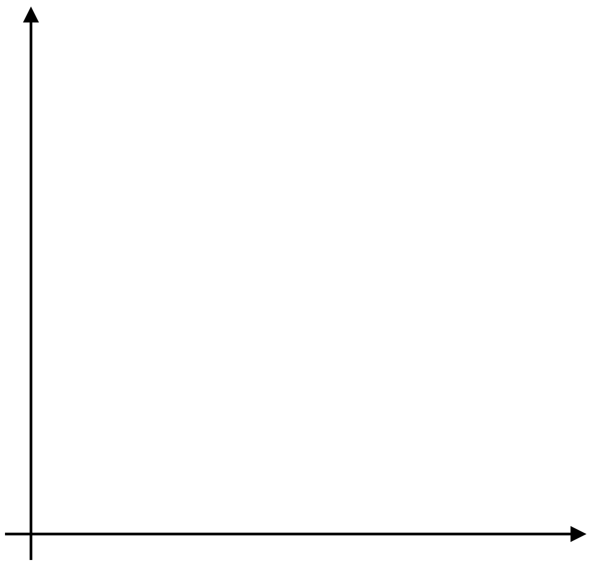
d) $\begin{aligned} 40x+20y&=440 \\ 30x+75y&=1050 \end{aligned}$

2) A furniture factory makes couches and chairs. Each couch needs 6 hours of construction time and 4 hours of upholstering time, while each chair needs 2 hours of construction and 3 hours of upholstering. Each day, the factory has available a total of 90 hours for construction and 85 hours for upholstering. Build an equation to represent construction hours, and a second for upholstering, and then determine the intersection of the two lines, which gives the number of couches and chairs built where we use up all of the available hours daily.

SECTION 1.4: LINEAR REGRESSION

Linear Regression is just one of MANY regression models used in statistical modelling of real data for the purpose of making predictions for unknown values based on known values. We will look at finding the Linear Regression Line both by use of formulas and also by use of our graphing calculators.

First, a simple example to work through, with the following x,y pairs: $(1,3),(3,4),(4,7),(7,8)$. Mark both axes by 1's and plot the four coordinate points below. Try to be very consistent with your tick-marks.



X	Y	X^2	X*Y
1	3		
3	4		
4	7		
7	8		

Now, try to draw a line through the plotted points, what is called a "Line of Best Fit" because it should be as close as possible to all of the points, and will not necessarily hit any of them exactly. Once we have an equation for the mathematically determined line, you can see how well you did.

Next, fill in the table above, where the X^2 column is the Square of the respective X-coordinate, and the X*Y column is the product of each X,Y pair. Then total up each column, in the bottom of each column.

The total in Column X is described in the formulas below by Σx, and similar for the other column totals. Use the formulas to first determine the Slope value, *m*, and then use *m* in the formula for *b*, the Y-Intercept.

$$m = \frac{N \cdot \Sigma x \cdot y - \Sigma x \cdot \Sigma y}{N \cdot \Sigma x^2 - (\Sigma x)^2} \qquad b = \frac{\Sigma y - m \cdot \Sigma x}{N}$$

where N = # of X,Y pairs

Did you get this? $y = 0.88x + 2.2$ Does your sketched line seem to agree?

Use the formulas $m = \frac{N \cdot \Sigma x \cdot y - \Sigma x \cdot \Sigma y}{N \cdot \Sigma x^2 - (\Sigma x)^2}$ *and* $b = \frac{\Sigma y - m \cdot \Sigma x}{N}$ on the below sets of values, rounding each of *m* and *b* to 2 decimal places, as necessary. BUT, keep 5 places(or the fraction) for *m* when plugging into the formula for *b*. The line for each is under the respective tables. Did you get these?

X	Y	X^2	X*Y
4	2		
7	4		
9	8		
12	11		
13	13		

$y = 1.24x - 3.57$

X	Y	X^2	X*Y
21.3	442		
33.7	576		
38.6	612		
44.5	729		
48.3	745		
53.8	854		

$y = 12.47x + 160.45$

Starts to get pretty unwieldy with more X,Y pairs and/or bigger numbers in the mix, doesn't it?
Some of the burden of handling these calculations can be placed onto our calculators. The recommended calculators are the TI-83 or TI-84, and so the instructions will be based on using those models. As said at the start of the semester, those choosing to use a TI-89 or nSpire or Casio models will need to locate the proper functionality on their machines.

1) Hit the "STAT" key and select the first option available, "EDIT". Enter the values of X into L1 and the values of Y into L2. To exit the EDIT screen, hit "2nd" and then "MODE"(so QUIT, above MODE).
2) Hit "STAT" again, arrow to the "CALC" column and select 2-Var Stats. The default usage of this function is lists 1 and 2, so you need not tell the calculator which lists. However, if you do use other lists than 1 and 2, you can place those list names after 2-Var Stats by hitting "2nd" and then "3" for L3, as an example. And use a Comma, the button directly above 7, to separate you list choices.

In 2-Var Stats, you should see the values for N, Σx , Σy , Σx^2 , *and* Σxy , all of the necessary values for use in the two formulas we have been using so far. Verify these values for each of the examples at the top of the page, or use this to see what went wrong if you did not get the correct lines in those examples.

Our calculators have an even more powerful tool, and computer software will have even more advanced usages than this. Those of you who will some day need to use these sorts of calculations and modelling techniques will surely do so on those advanced software packages, as opposed to filling out tables and using formulas like we did above. So, now we look at what the TI's offer us in this way.

Take the previous examples X,Y sets and again place them into the STAT EDIT screen as lists L1 and L2. Remember to use 2nd/MODE(Quit) to exit the screen. Then, once again, hit STAT and arrow over to CALC. Now arrow down to LinReg(ax+b) and select this function. Just like with 2-Var Stats, the default usage is for L1 and L2, but can be used for any lists as before by placing their names after the basic command.

Verify that you get the same lines as we found by using the formulas for each of the above examples.

Now, we use the lines to make predictions.

i) Take the line $y = 1.24x - 3.57$ and suppose we wish to explore having an "input" value of $x = 10$(which notice was not in the table we used to find the line). What does the line predict for y?

On the other hand, what if we have a goal for y, say $y = 16$. What x would be necessary according to our line?

ii) now use the line from the second table, $y = 12.47x + 160.45$

a) If a target of 700 in output is desired, what input is needed?

b) An input of 51.2 has been discovered, what would the line predict for an output?

SECTION 2.1: SOLVING SYSTEMS OF EQUATIONS WITH A UNIQUE SOLUTION

In Chapter 1 we took a look at finding the intersection point of two lines on a graph. Chapter 2 begins with a look at a more formal approach to this same idea. The process is going to consist of using 3 operations. Using the system below, do each of the described operations, where each time you will still have 2 equations:

$$\begin{aligned} 2x+5y&=18 \\ x+3y&=10 \end{aligned}$$

1) Swap the 2 equations' positions, top-to-bottom and bottom-to-top.

2) Multiply both sides of the SECOND equation by 2.

3) The third operation involves adding a non-zero multiple of one equation to another equation. The purpose of this is to ***eliminate*** the variable in the "added to" equation. For example, you should have the system below after performing the first two operations, and we would now like to eliminate the *4x* term in the bottom equation. So, we add -4 times the top equation to the bottom one. NOTE: the top equation DOES NOT get changed, we just use it for our stated purpose.

$$\begin{aligned} x+3y&=10 \\ 4x+10y&=36 \end{aligned}$$

If you look at each of the operations, you will hopefully notice that it was just the Coefficients of *x* and *y*, as well as the right-side constants, that change. But the "columns" always represented *x, y*, and the constants. To simplify our work, we place just those coefficients and constants into a matrix. Going back to our original system:

$$\left.\begin{aligned} 2x+5y&=18 \\ x+3y&=10 \end{aligned}\right\} \qquad \left[\begin{array}{cc|c} 2 & 5 & 18 \\ 1 & 3 & 10 \end{array}\right]$$

Perform the same three operations as above on the values in the matrix, where some notation is given as a shorthand way to describe each operation:

1) $R_1 \leftrightarrow R_2$ 2) $2 \cdot R_2 \to R_2$ 3) $-4R_1 + R_2 \to R_2$

You should now have this partially reduced matrix: $\left[\begin{array}{cc|c} 1 & 3 & 10 \\ 0 & -2 & -4 \end{array}\right]$. The 1 in the top left corner is referred to as a "Leading 1" since it is the first non-Zero value in its Row. We would like to now move down a row from this leading 1, and then to the right and make that matrix element another leading 1. And once it is a leading 1, we will empty all other values in its column. Do so now, and write out notations to reflect each of the operations(Hint: it should take 2 op's) you perform.

Do you have this matrix? $\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right]$ Now, let us recall that the x-coefficients were placed in the first column and the y-coefficients into the second column. Those columns still, of course, represent those same variables. Turn each row of this final matrix back into its respective equation, and we have a unique solution to our original system of equations: x = 4 and y = 2

EXERCISE: Use the same process, in matrix form, including notations for each step, to solve:

$$\begin{aligned} 3x + 4y &= 25 \\ x + 6y &= 20 \end{aligned}$$

Did you get the solution (5 , 5/2) ??

PRACTICING ELEMENTARY ROW OPERATIONS:

$$\begin{bmatrix} 3 & -1 & 5 & -4 \\ 8 & 4 & -6 & 2 \\ -1 & 2 & -3 & 5 \end{bmatrix}$$ **Use this original Matrix to perform each Row Operation**

i) $R_2 \leftrightarrow R_3$

ii) $-4R_3 \rightarrow R_3$

iii) $3R_1 + R_2 \rightarrow R_2$

iv) $-2R_3 + R_1 \rightarrow R_1$

PIVOT: a "Pivot" is where at a chosen element in a matrix, we make that element a +1 and then proceed to use that 1 to eliminate all other values in its column(above or below). Perform a Pivot on the 2 in the 2^{nd} Row and 4^{th} column of the matrix in the previous exercise.

$$\begin{bmatrix} 3 & -1 & 5 & -4 \\ 8 & 4 & -6 & 2 \\ -1 & 2 & -3 & 5 \end{bmatrix}$$

EXERCISES: Use the process to reduce each system, turned into matrix form, and find the Unique solution to each.

a) $\begin{aligned} 2x + y + 2z &= -1 \\ 3x + 2y + 3z &= -3 \\ x + y + 2z &= -3 \end{aligned}$

b) $\begin{aligned} 4x + y + z &= 3 \\ 2x + y + z &= 2 \\ 3x + y + 2z &= 1 \end{aligned}$

SECTION 2.2: INFINITE SOLUTIONS AND NO SOLUTION

In Section 2.1 we saw that a System of Equations can have a Unique Solution of the form (x,y) or (x,y,z), and in fact can have even more variables and still have such a solution. Now, though, we will look at solving Systems of Equations which do not have such unique solutions.

Convert the system below into matrix form and follow the same process as we used in Section 2.1, where you should establish a "leading 1" in the top left corner, followed by another in Row 2, Column 2. Do so and then stop there.

$$\begin{aligned} 2x + y + 3z &= 8 \\ -x - z &= -3 \\ x + y + z &= 8 \end{aligned}$$

Do you get this: $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right]$?? What does the bottom Row suggest back in equation form?

We have a similar result to a situation we saw in Section 1.3, and the same conclusion of "No Solution"

Now try this System of Equations:

$$\begin{aligned} 3x + y + z - 7w &= 2 \\ x + y - 2w &= -3 \\ x + y + z + w &= -2 \\ 2x + y + z - 3w &= 4 \end{aligned}$$

Again, you find a Row suggesting a "false" equation, and we again conclude No Solution

Now try this System, where it is slightly different than the first example on the previous page, but the result is quite different.

$$\begin{aligned} 2x + y + 3z &= 8 \\ -x - z &= -3 \\ x + y + z &= 5 \end{aligned}$$

You should get this: $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The Row of all 0's is actually not a problem, and can now be ignored. But take each of the first two Rows and convert them back into Equations with their proper variables(x, y, and z) back where they belong. $x + 2z = 3$ should be converted to $x = -2z + 3$ and $y - z = 2$ should be converted to $y = z + 2$. Now, because we used *z* to describe both *x* and *y*, *z* can be "Any Real Number". This is an "Infinite Solution" because *z* has an infinite number of choices we can give it.

Now, reduce this system and find its properly represented Infinite Solution:
USE ROW OPERATIONS, NOT "RREF"

$$\begin{aligned} 3x + y + z - 7w &= 2 \\ x + y - 2w &= -3 \\ x + y + z + w &= -2 \\ 2x + y + z - 3w &= 0 \end{aligned}$$

Convert the following System to a Matrix and reduce it to determine its Infinite Solution properly represented. "RREF" can be used

$$\begin{aligned} 2x + y + 5z &= -1 \\ x + y + 2z &= 1 \\ 3x + y + 8z &= -3 \end{aligned}$$

Did you get this: $\begin{cases} x = -3z - 2 \\ y = -z + 3 \\ z = \text{Any Real} \end{cases}$

Let's explore some possibilities.

Start with $z = 3$. Find the associated values for *x* and *y*. Check this solution in each of the 3 original equations to verify it is a valid solution.

Do the same where $z = -8$. Does the (x,y,z) ordered triple "check" in all 3 equations again?

Now try to find the particular solution if $y = -2$. Again, check it to verify it is indeed valid.

EXERCISES: Use "RREF" to solve each System of Equations. Show each reduced matrix and a PROPERLY STATED solution.

a) $$\begin{aligned} 2x + 6y + z &= 10 \\ x + 3y &= 6 \\ 3x + 9y + z &= 16 \end{aligned}$$

b) $$\begin{aligned} 2x + 2y + z &= 9 \\ x + 2y + z &= 5 \\ 2x + 5y + 3z &= 9 \end{aligned}$$

c) $$\begin{aligned} 5x + 10y + 4z + 4w &= -7 \\ x + 2y + z + w &= -1 \\ -x - 2y - 2z - w &= -3 \\ 2x + 4y + 2z + 2w &= -2 \end{aligned}$$

SECTION 2.3: OPERATIONS ON MATRICES

Do the following Matrix operations WITHOUT USING YOUR CALCULATORS

$$A=\begin{bmatrix} 2 & -3 \\ -1 & 6 \end{bmatrix} \qquad B=\begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \qquad C=\begin{bmatrix} -3 & 5 \\ 4 & -2 \end{bmatrix}$$

i) $4A$

ii) $A+C$

iii) $3B+2C$

iv) $2A-C+4B$

EXERCISE: Find the values of Matrix E to make the equation true: $4F=2E+D$

$$D=\begin{bmatrix} -2 & -8 \\ 6 & 4 \end{bmatrix} \qquad E=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad F=\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}$$

MATRIX MULTIPLICATION

First, some practice on the Dot Product process. Multiply the matrices BY HAND

a) $\begin{bmatrix} 2 & -1 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ -6 \end{bmatrix} = [(\quad)(\quad)+(\quad)(\quad)+(\quad)(\quad)] = [\qquad]$

b) $\begin{bmatrix} 5 & 10 & 7 \end{bmatrix}\begin{bmatrix} -6 \\ 5 \\ -8 \end{bmatrix} = [(\quad)(\quad)+(\quad)(\quad)+(\quad)(\quad)] = [\qquad]$

Now, STILL BY HAND, multiply the following matrices, writing out each Dot Product in their respective slots in the extended matrix. Then find the simplified final product matrix.

$$\begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} 8 & 1 & 6 \\ 0 & 7 & 4 \end{bmatrix} = \left[\begin{array}{c|c|c} \qquad & \qquad & \qquad \\ \hline \qquad & \qquad & \qquad \end{array}\right] = \begin{bmatrix} \qquad & \qquad & \qquad \\ & & \end{bmatrix}$$

SIZE/DIMENSIONS OF MATRICES and whether we can Add/Multiply matrices. Recall from lecture that we can only add matrices of exact same dimensions, the "sum matrix" being again these same dimensions. We can only multiply matrices where the column dimension of the first equals the row dimension of the second, with the "product matrix" having the row dimension of the first and column dimension of the second. We modeled the product in this fashion: $\underset{M\,x\,N}{A} \cdot \underset{N\,x\,P}{B} = \underset{M\,x\,P}{C}$. Given the below set of matrices, decide which calculations can or cannot be done according to proper dimension rules.

$$A = \begin{bmatrix} 1 & 9 \\ 8 & 2 \\ 3 & 7 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix},\ C = \begin{bmatrix} 7 \\ 4 \end{bmatrix},\ D = \begin{bmatrix} 2 & 8 \\ 6 & 4 \end{bmatrix},\ E = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 9 & 7 \\ 5 & 3 & 1 \end{bmatrix},\ F = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix},\ G = \begin{bmatrix} 5 & 4 & 3 \\ 2 & 1 & 0 \end{bmatrix}$$

For those that CAN be done, determine what are the dimensions of the final matrix.

i) $D+E+F$

ii) $AG-2E$

iii) $GE-AD$

iv) $CBE+G$

v) $DGAFE$

vi) $BA+CD$

EXERCISE: The matrix below lists 6 Math 125 students and the MyMathLab Percent averages they earned one semester on Homework, Quizzes, Midterms, and the Final Exam. The course webpage says homework is worth 80 points, quizzes 120, midterms 200, and the final exam 200, so a grand total of 600 points for the semester.

$$\begin{array}{r} \\ Amy \\ Bob \\ Chet \\ Dave \\ Eva \\ Fiona \end{array}\begin{array}{c} \begin{array}{cccc} HW & Qz & Mdtm & Fin \end{array} \\ \begin{bmatrix} 95.7 & 82.4 & 77.2 & 80 \\ 56.8 & 84.8 & 73.6 & 75 \\ 86.1 & 93.2 & 87.5 & 90 \\ 76.3 & 94.1 & 72.8 & 85 \\ 99.4 & 73.6 & 70.2 & 75 \\ 96.8 & 92.4 & 78.2 & 70 \end{bmatrix} \end{array}$$

Construct any other necessary matrices(appropriately labeled) and give a Matrix expression(Hint: scalar multiplication might be needed) that will give each student their total points for the semester. How many points out of 600 maximum did each earn?

SECTION 2.4: THE INVERSE OF A MATRIX PLUS APPLICATION

It was described in class that two Square Matrices, A and B, are Inverses of one another if $AB = BA = I$, where I is the appropriately sized Identity Matrix.

First, as a demonstration of the property, verify that the two matrices below are in fact Inverses of one another, by multiplying them together in both orders, AB *and* BA. FOR GOOD PRACTICE, DO THIS BY HAND, WRITING OUT EACH DOT PRODUCT FULLY.

$A = \begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 3 \\ 7 & -5 \end{bmatrix}$

EXERCISE: Decide whether each pair below are/are not Inverses, still using the concept: $AB = BA = I$
To speed up your work, use calculators this time.

i) $A = \begin{bmatrix} 4 & 3 \\ 6 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 5/2 & -3/2 \\ -3 & 2 \end{bmatrix}$

ii) $A = \begin{bmatrix} 16 & 6 \\ 10 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -5/2 \\ -3/2 & 4 \end{bmatrix}$

iii) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 3 & -1 \end{bmatrix}$

FINDING THE INVERSE OF A 2x2 MATRIX BY USE OF A FORMULA

In class, the below formula was given for finding the Inverse of a 2x2 Matrix. Use it to find the Inverse of each matrix. **It is highly recommended you determine *ad-bc* first**.

$$if \quad A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad , then \;\; A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \; , \; ad-bc\neq 0$$

i) $\begin{bmatrix} 4 & 3 \\ 6 & 5 \end{bmatrix}$

ii) $\begin{bmatrix} 16 & 6 \\ 10 & 4 \end{bmatrix}$

iii) $\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$

iv) $\begin{bmatrix} -2 & 4 \\ 0 & M \end{bmatrix} \; M\neq 0$

v) $\begin{bmatrix} -2 & 3/2 \\ 3/2 & -1 \end{bmatrix}$

AN APPLICATION FOR USING THE INVERSE

Earlier in the chapter, we explored solving Systems of Linear Equations and solved them by use of Gaussian Elimination. We have now shown in class that if there is a Unique Solution, it can be found by converting the System of Equations into Matrices *A, X, and B*. The matrix Equation $AX = B$ represents the actual system, and it was shown that *X*, our desired solution, could be found by $X = A^{-1}B$

Given the system $\begin{matrix} 7x + 3y = 23 \\ 9x + 4y = 30 \end{matrix}$, deconstruct the system into matrices *A, X, and B.*

$A = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \qquad X = \begin{bmatrix} \quad \\ \quad \end{bmatrix} \qquad B = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$

Now, find the Inverse to A, by hand using the formula:

Next, determine the values in X by use of $X = A^{-1}B$, performing the calculations BY HAND.

Finally, check your solution back in the original equations to verify it is the correct Unique Solution.

EXERCISE: Redo the above process on this system: $\begin{matrix} 4x + 3y = 13 \\ 8x + 5y = 25 \end{matrix}$

SECTION 2.5: GAUSS-JORDAN METHOD OF FINDING THE INVERSE

The Gauss-Jordan method for finding an Inverse to a Matrix is quite similar to the Gaussian Elimination process we used for solving a System of Equations earlier in the chapter. All our decision in Gaussian Elimination were made based on the values to the left of our "Augment Bar", and exactly the same thing will occur in Gauss-Jordan. Use Gauss-Jordan process to find the Inverse of A, the left side of the matrix already augmented below with an appropriate sized Identity Matrix.

Write out proper notations of each Elementary Row Operation and show all intermediate matrices. Once finished, verify by use of your calculators. You will not necessarily need all of the provided matrices.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Now that we have found the Inverse to $\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \\ 2 & -1 & 3 \end{bmatrix}$, specifically finding $\begin{bmatrix} 5 & -7 & -2 \\ 4 & -5 & -2 \\ -2 & 3 & 1 \end{bmatrix}$, use your calculators to verify that multiplying them in either order does give you a 3x3 Identity matrix.

Next, suppose you are asked to solve the below Systems of Equations WITHOUT USING YOUR CALCULATORS. Apply the concept from Section 2.4: $X = A^{-1}B$ by appropriately deciding which of the matrices is A and which is A^{-1}

i) $\begin{aligned} x + y + 4z &= 18 \\ y + 2z &= 10 \\ 2x - y + 3z &= 9 \end{aligned}$

ii) $\begin{aligned} 5x - 7y - 2z &= 38 \\ 4x - 5y - 2z &= 28 \\ -2x + 3y + z &= -15 \end{aligned}$

EXERCISE: Use Gauss-Jordan to find the inverse of $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$

SECTION 2.6: LEONTIEFF INPUT-OUTPUT ANALYSIS

Discussed in lecture was the idea of wanting to determine how much total production from various industries, or divisions of a company, would be necessary to not only satisfy the internally used output, but also have enough to satisfy the needs of the public as well.

The Input-Output values were placed into a square matrix, *A*, where we must label the matrix both across the top and down the side in exactly the same order. The industry listings on top refer to the one "Producing $1 of Output", while the listing on the side of the matrix refer to the industry where "Input is needed from".

For the problem presented, first construct and fully label a matrix, and then fill in the appropriate values. WARNING: you should never assume information is given "in order".

A small country has two industries, Bacon and Eggs. To produce $1 of bacon, that industry needs $0.10 of their own output and $0.20 from the eggs industry, while to produce $1 of eggs, that industry needs $0.10 of their own output and $0.30 from the bacon industry.

$A = \begin{bmatrix} & \\ & \end{bmatrix}$ and now $I - A = \begin{bmatrix} & \\ & \end{bmatrix}$

Next, use the formula from back in Section 2.4 to find the Inverse: $(I - A)^{-1}$ in FRACTIONS

Now suppose we are told that the population needs $36000 in bacon and $30000 in eggs. Build an appropriately labeled matrix *D* for these values. Then determine total production, $X = (I - A)^{-1} D$

What is the economy has more than 2 industries? The Inverse Matrix needed is far too complicated for us to determine by the methods we have studied, the formula for a 2x2 Inverse won't work, and the gauss-Jordan process would force us to work with incredibly large numbers in the numerator and denominators of fractions. So, this example will mirror what MML demands of you in determining total production values.

An economy consists of three industries: metals, plastic, and energy. To produce $1 of metal, that industry needs $0.12 of their own output, $0.02 from plastic, and $0.25 from energy. To produce $1 of output, the plastics industry needs $0.08 of their own output, $0.15 from metals, and $0.18 from energy. To produce $1 of output, the energy industry needs $0.05 of their own output and $0.09 from the metals industry. The public needs each year $65 million in metals, $48 million in plastic, and $82 million in energy. How much must each industry produce in total to satisfy this demand?

a) Build a fully labeled Input-Output matrix, *A*. Make sure you line up properly which industry is producing $1 of output to which industry is providing a specified amount of input.

b) Use your calculator to calculate $(I-A)^{-1}$. The command should look like this: $(identity(3)-[A])^{-1}$

Now you need to round each of the decimal values in $(I-A)^{-1}$ to 2-decimal places. The best way to do this is save the result into Matrix [B] using the "store" key on the calculator. Then go into "Edit" on Matrix *B* and round each value you see down to 2-decimal places. Write the "rounded" matrix below.

c) Build and label matrix *D*, the public demand matrix. USE the given values, "in millions", NOT full length values with all of the 0's. So, for example, 65, **NOT 65000000**. Now multiply the rounded-off matrix with *D*. Do you get the below values in *X*?

$$\begin{bmatrix} 95.15 \\ 54.27 \\ 121.19 \end{bmatrix}$$

If you did not get this, check that *A* was properly built as well as your rounding in $(I-A)^{-1}$.

Question: Would we ever want to look this problem from the industry end of the situation? Meaning, what if we knew how much total production, *X*, is possible, and want to then see what is actually available for public consumption? Recall that we started the whole process by assuming total production minus a calculation of "internal industry usage" would leave an amount, *D*, for the public. In that discussion, we created the equation $X - AX = D$, which was then converted into $X = (I - A)^{-1} D$, the calculation we have used in the above examples. Let us look at an example using $X - AX = D$.

Olympia Island has 3 industries, Gold, Silver, and Bronze. To produce $1 of Gold, that industry needs $0.03 of gold, $0.06 of silver, and $0.10 of bronze. To produce $1 of Silver, that industry needs $0.07 of gold, $0.09 of silver, and no bronze. To produce $1 of Bronze, that industry needs $0.02 of gold, $0.12 of silver, and $0.08 of bronze. Build *A*, the Input-Output matrix.

a) Suppose the three industries have the capacities to produce only $60000 in gold, $50000 in silver, and $75000 in bronze. Use $X - AX = D$ to determine how much of this total output would be left over for the public.

b) How much of that total production is then used internally by the three industries?

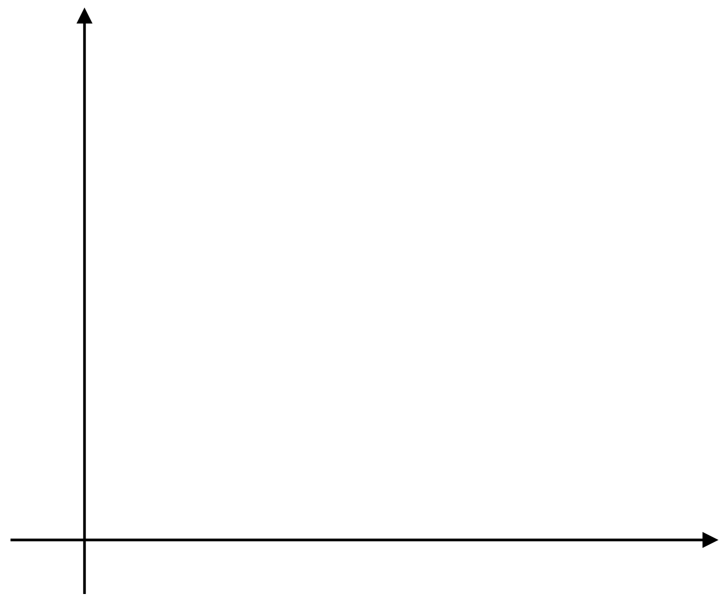
c) Now suppose we do know the public demand figures of $90000 in gold, $82000 in silver, and $68000 in bronze. Use *A* as created above and the methods in the example on the previous page, including the 2-decimal rounding of $(I - A)^{-1}$, to determine what total production, *X*, is necessary.

SECTION 3.1: GRAPHING LINEAR INEQUALITIES AND FEASIBLE REGIONS

We start with a reminder of the smart way to graph a Linear Equation for the typical example we see in this course, namely using BOTH X- and Y-Intercepts, when available.

EXAMPLE: Graph the Line $8x+6y=240$

a) Determine the Coordinates of BOTH intercepts
b) Choose reasonable values to mark the X- and Y-Axes
c) Plot and Label each intercept with their coordinates
d) Sketch the Line



Now, suppose instead of just a Linear Equation, we have a Linear Inequality: $8x+6y \le 240$
In Algebra coursework, you should have been taught to shade one side or the other, where you would choose the "good side", as this represented the half of the plane satisfying the Inequality.

However, in Math 125, we need to shade the "bad side" of our graphed line, so that when we do this on multiple inequalities, we will leave blank an area that is still "good" for every one of those inequalities.

Which side of the above graphed line should you shade? To decide, select a Coordinate Point that is NOT on the line itself. Try both $(0,0)$ *and* $(30,30)$ by plugging them into $8x+6y \le 240$. Which one leaves you with a true Inequality statement? Shade the side opposite the "good point", which is then shading the side with the "bad point".

On the SAME axes above, now graph using all of the same steps: $10x+20y \ge 200$

HINT: quite often, $(0,0)$ is available as a "test point" and as a great choice when this is so

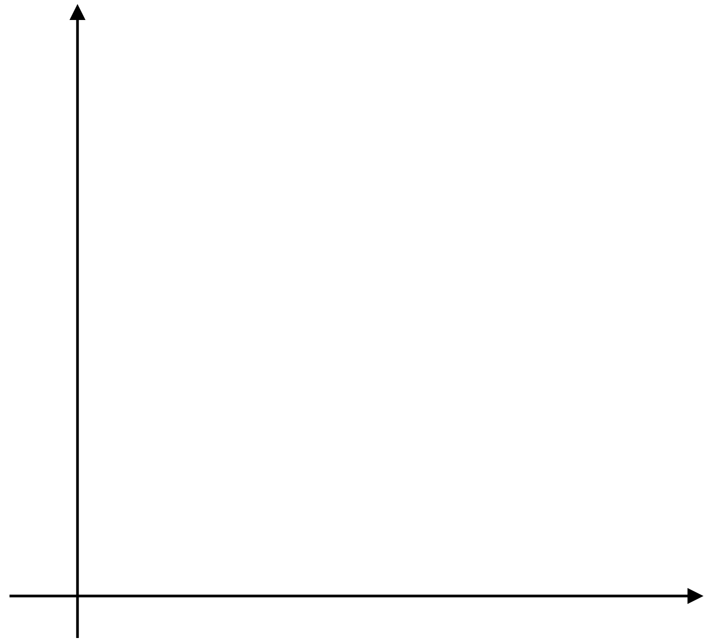
FEASIBLE REGIONS: Also called Feasible Sets(and thus abbreviated as FS) are the region on a graph where multiple Inequalities "agree", meaning that all of them can be satisfied by every coordinate pair this FS represents, both its boundary line segments(or half-lines) as well as all points in its interior.

GRAPH: place all three on the same axes

$5x + 6y \le 300$

$x \ge 18$

$y \ge 10$



Once all 3 have been shaded, do you have an unshaded triangle? That is the FS for these Inequalities.

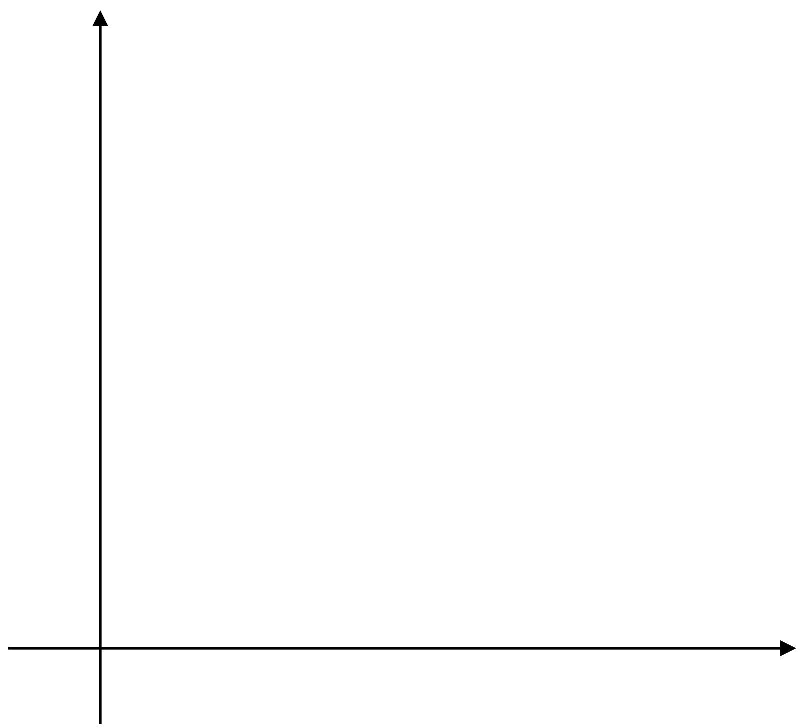
Now that we have a well-defined FS, determine the Coordinates of each of the 3 "corners" of the FS.

EXERCISE: Graph the below Inequalities on a single pair of axes, and then determine the coordinates of each corner of the FS once it is completed.

$2x + 6y \le 90$

$5x + 3y \le 105$

$x \ge 0$, $y \ge 0$

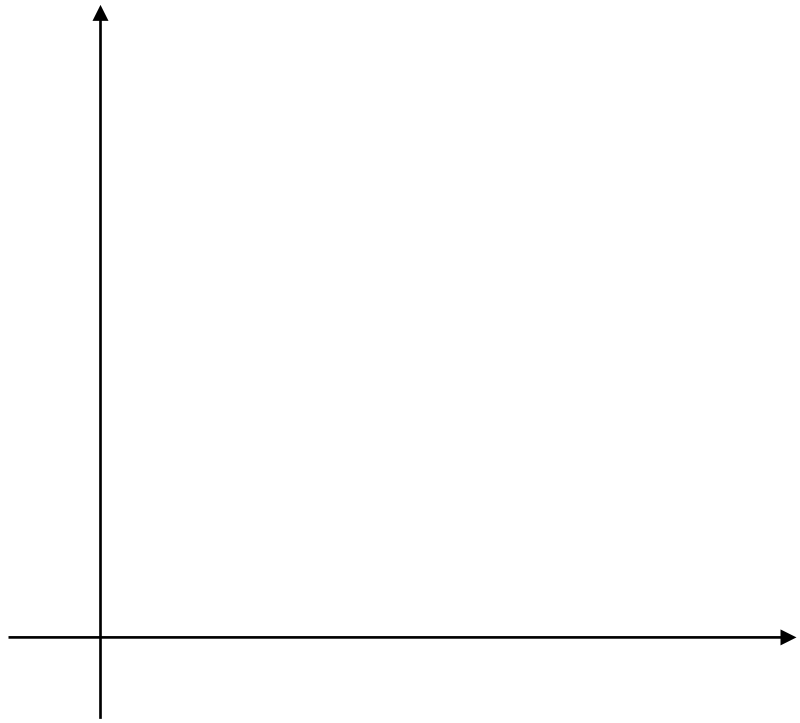


EXERCISE: Graph the below Inequalities on a single pair of axes, and then determine the coordinates of each corner of the FS once it is completed.

$x + 4y \leq 40$

$3x + 4y \leq 120$

$x \geq 12$

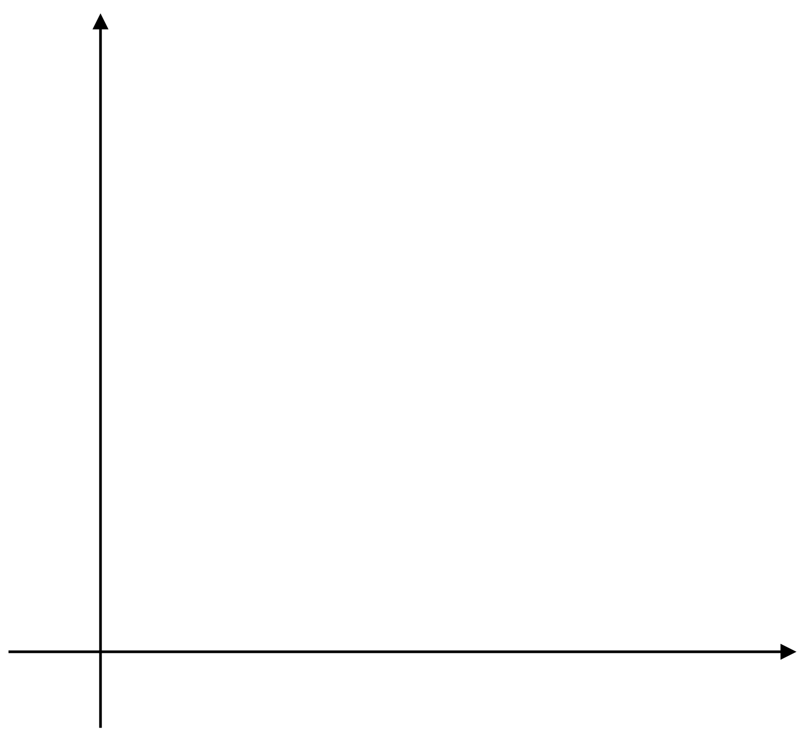


EXERCISE: Graph the below Inequalities on a single pair of axes, and then determine the coordinates of each corner of the FS once it is completed.

$3x + 2y \geq 48$

$3x + 4y \geq 72$

$x \geq 0$, $y \geq 0$



SECTION 3.2: THE LINEAR PROGRAMMING PROBLEM

As described in lecture, a *Linear Programming Problem* consists of three main components: an Objective Function, a set of Constraints, and Non-Negativity Constraints. This section is mainly for the purpose of converting applied situations(word problems!!) into properly constructed Linear Programming Problems(LP problems from now on). Sometimes, but NOT always, a table setup provides us a useful structure for laying out the information that easily translates into the constraints and objective function. We begin with such examples.

EXAMPLE: WidgetWorld builds two products, a basic widget and a deluxe widget. Each basic widget needs 2 hours of assembly and 1 hour of painting, and each deluxe widget needs 3 hours of assembly and 4 hours of painting. The workroom has available each day 120 hours for assembly and 120 hours for painting. Profits are $12 for each basic model and $15 for each deluxe model. Construct a full Linear Programming Problem to help WidgetWorld earn the greatest profits on a daily basis.

i) **Name your variables and what they refer to**. Identify what quantities in the application are unknown and will specifically be determined. Here it is how many basic widgets and how many deluxe widgets. So:

x = # ___________________ and y = #___________________

Use the topics described by our variables as the headings for the columns in the table below, and then list the "inputs" that go into these items as the row headings. Then fill in the table with appropriate values. Often, we put the "objective" in the bottom row, here that is Profits.

Inputs ∨	X=	Y=	Available
Profit			

ii) **Build a proper Objective Function**. Often, the aspects of this are given at the end of the application write-up, yet will be the first line of the LP Problem. **Always** include whether we are wanting to *Minimize* or *Maximize* in achieving our goal.

iii) **Construct all necessary Constraints**, including the **Non-Negativity Constraints**. In the application wording will be something to suggest whether we need go only up to, but not over, some value(perhaps the word "available") or we need to reach at least some, or go above(perhaps a "required" amount). This will help you decide which to use: $\leq$ *or* $\geq$

EXERCISE: Construct the LP-Problem for the following situation.

A national organization wants to increase their membership and is hoping to catch the attention of younger adults. When they do a college campus visit, it costs $100, takes 4 person-hours, and generates 32 new members. When they visit a shopping mall, it costs $40, takes 3 person-hours, and generates 18 new members. For this year's membership drive, they have budgeted $8000 and have 390 person-hours available. How many campus visits and mall visits should the organization make to obtain the greatest amount of new members, and how many new members will they enroll?

i) **Name your variables and what they refer to**. Then fill in the table below

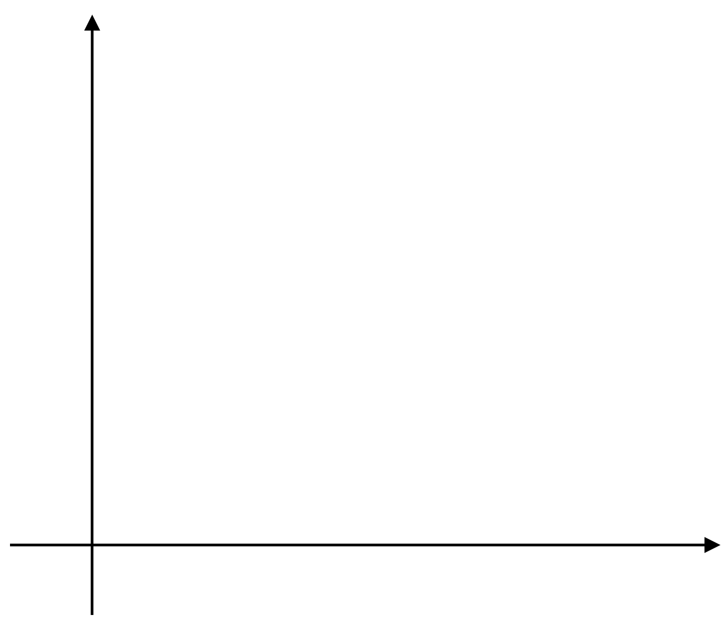
x = # ____________________ and y = #____________________

Inputs ∨	X=	Y=	Available

ii) **Build a proper Objective Function**.

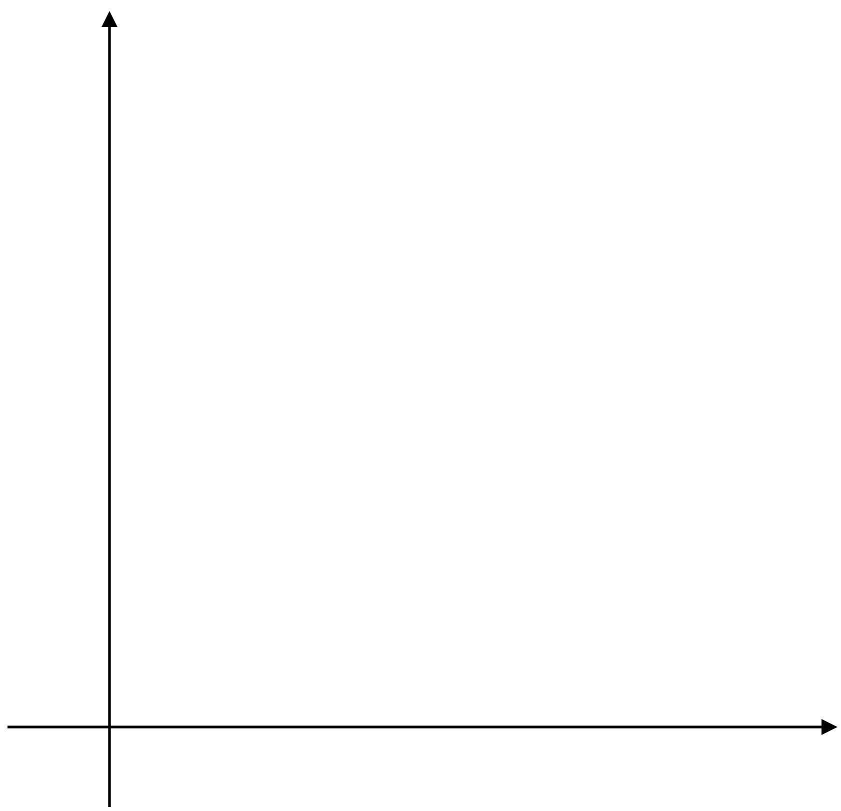
iii) **Construct all necessary Constraints**, including the **Non-Negativity Constraints**.

iv) Graph the Feasible Set and determine the coordinates of ALL corner points.



EXERCISE: EnergeeBar is designing a new breakfast bar and plans to use two natural ingredients in each bar. Ingredient A has 3g of protein per ounce, 4g of fat per ounce, 6 units of vitamin B_{12}, and costs \$0.12 per ounce, while Ingredient B has 2g of protein per ounce, 6g of fat per ounce, 2 units of vitamin B_{12}, and costs \$0.08 per ounce. The nutritionist wants at least 60g of protein, 60mg of fat, and 84 units of B_{12} per bar. How many ounces of each ingredient should be used to create a breakfast bar meeting the nutritional requirements at least cost?

Build a properly constructed LP-Problem. Then graph the problem and determine the coordinates of all corner points of the Feasible Set. Use of a table is not required, but is still recommended.



EXERCISE: ScrappleLot company owns two scrap metal plants. Plant I produces 3 tons of iron, 1 ton of copper, 2 tons of aluminum daily, at a cost of $600, while Plant II produces 2 tons of iron, 2 tons of copper, 1 ton of aluminum daily, at a cost of $800. The general manager wants Plant II to be run at least twice as many days as Plant I in filling an order received for 24 tons of iron, 20 tons of copper, and 12 tons of aluminum. How many days should ScrappleLot run each plant fulfill the order at least cost, and what is the least cost?

Build a full LP-Problem. A table is not required, but is still recommended. DO NOT GRAPH.

QUESTION: Do you have an Inequality constraint for the line that says: "The general manager wants Plant II to be run at least twice as many days as Plant I"? Did a table help to create this? Most likely, the table did not help, this being a perfect example of why tables are helpful only sometimes.

To construct a proper constraint for this line, start by setting up the basic comparison, Plant II to Plant I:

Plant II = y "versus" Plant I = x

Which should be "multiplied by 2"? If you are not sure, try a specific number for Plant I, say 10. If Plant I is to be run 10 days, what does the line suggest is necessary for plant II? If at least twice as many, then it must be run 20 or more, correct? Does that not mean we need to multiply Plant I, so *x*, by 2? Therefore, we now have:

$$y \qquad 2x$$

Now, since we want Plant II "at least" that much, we have this: $y \geq 2x$. Other acceptable forms of this same constraint would be these: $-2x + y \geq 0$ *or* $2x - y \leq 0$

SECTION 3.3: SOLVING THE LP-PROBLEMS BY USE OF THE FUNDAMENTAL THEOREM

The Fundamental Theorem of Linear Programming tells us that a Convex, Linear-based Feasible Set will realize an Optimal value to a Linear Objective Function, either at a corner of the Feasible Set or at adjacent corners along with the line segment connecting those adjacent corners. Thus, our graphing solution process to LP-Problems can be summarized as follows:

i) Construct the full LP-Problem from the application/word problem
ii) Sketch the graph of all constraints, shading properly to determine the Feasible Set
iii) Determine the coordinates of ALL corners of the Feasible Set. List them.
iv) Plug each of the corners into the Objective Function, and identify the desired Maximum or Minimum value, as stated in the Objective Function. The values of *x* and *y* are also useful in applications.

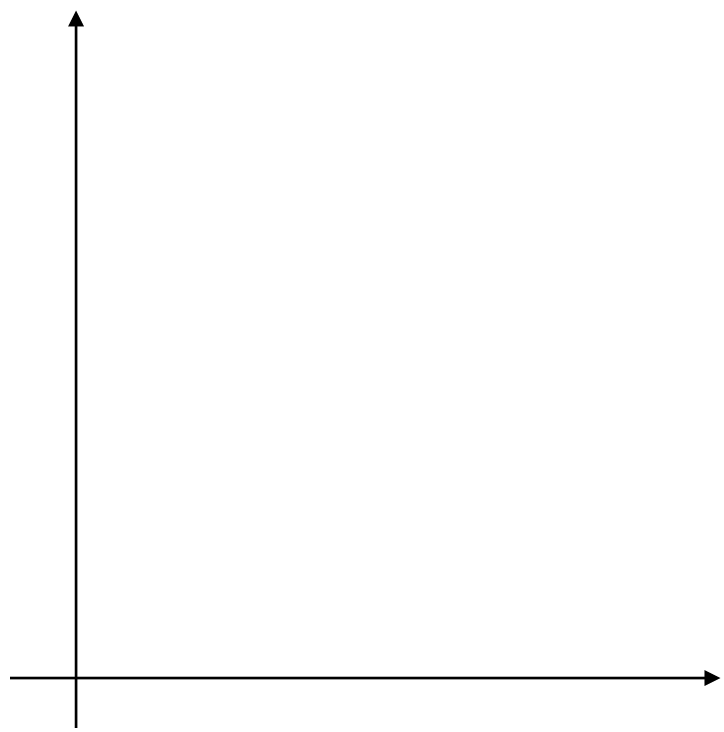
EXAMPLE: The below LP-Problem should be what you constructed for the WidgetWorld example at the start of the 3.2 worksheet. Follow the steps above to find the Maximum Profit, and also determine how many of each type of widget, basic and deluxe, should be made to realize this level of profit.

$Maximize: \ 12x + 15y \quad s.t.$

$2x + 3y \leq 120$

$x + 4y \leq 120$

$x \geq 0 \ , \ y \geq 0$



Max Profit: ___________ , # Basic Widgets = _________ , # Deluxe Widgets = __________

EXERCISE: For a fully *bounded* Feasible Set, the Fundamental Theorem says we can find BOTH a Minimum and a Maximum for a given Objective Function. Graph the below set of constraints, and obtain the list of ALL corners of the Feasible Set.

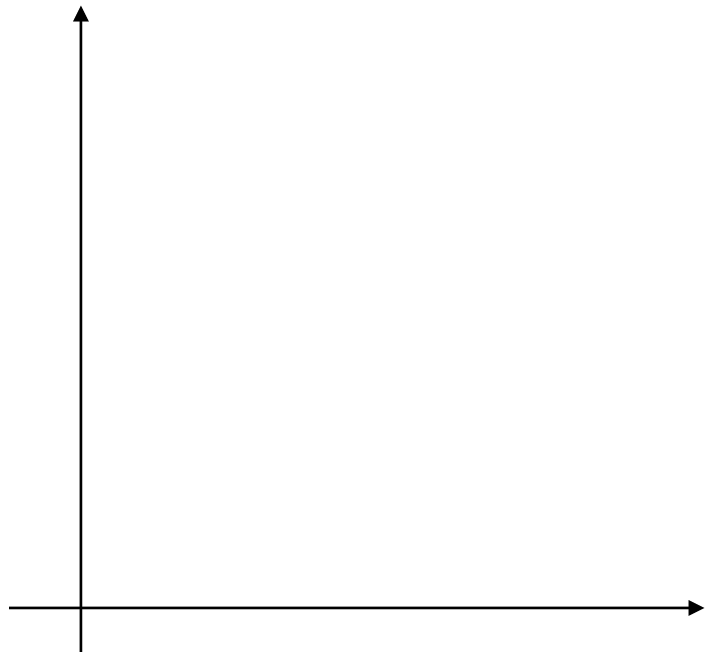
$x + y \le 800$

$x + y \ge 250$

$x \le 600$

$y \le 500$

$x \ge 0$, $y \ge 0$



Now suppose we have the Objective Function: $5000 - 12x + 15y$. Plus all of the corners into the function and find BOTH the minimum and maximum values.

What if instead, the Objective Function is this: $5000 + 18x - 14y$? You should get both a maximum and minimum again, but they are different values, and at different corners.

One Feasible Set will provide different solutions, all dependent on the Objective Function.

EXERCISE: Graph the Feasible Set, determine all corners, and find the minimum

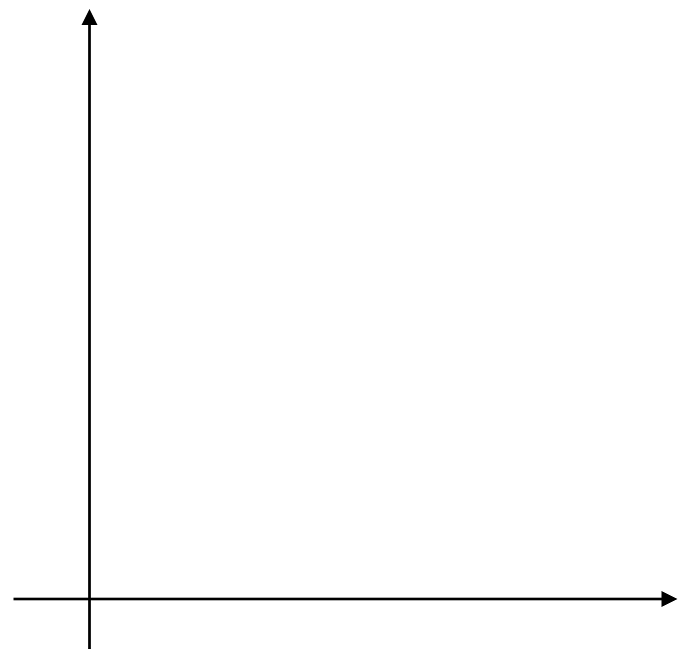
$Minimize: \ 18x + 30y \ \ s.t.$

$3x + 2y \geq 24$

$5x + 4y \geq 46$

$4x + 9y \geq 60$

$x \geq 0 \ , \ y \geq 0$



EXERCISE: Graph the Feasible Set, determine all corners, and find the maximum

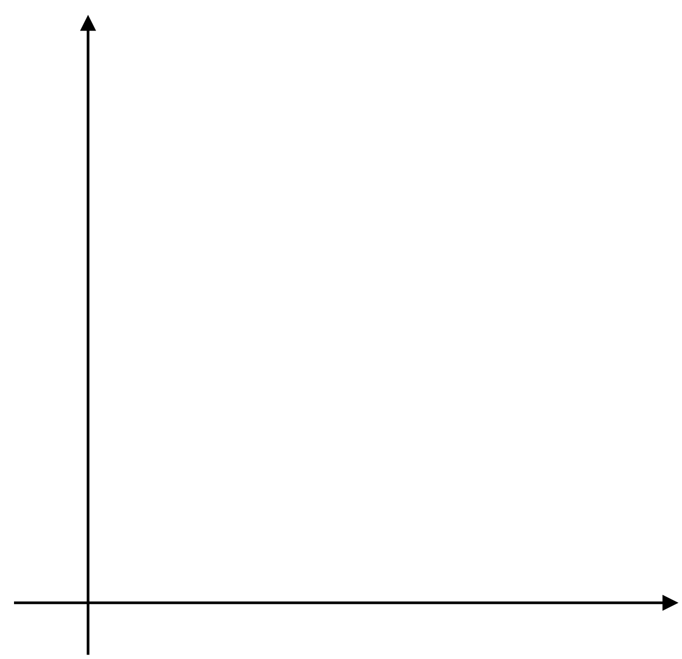
$Maximize: \ 18x + 9y \ \ s.t.$

$x + 5y \leq 60$

$x + y \leq 16$

$2x + y \leq 26$

$x \geq 0 \ , \ y \geq 0$

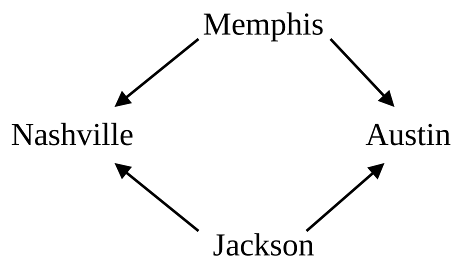


SECTION 3.4: SOLVING LP-PROBLEMS WITH 3 OR 4 UNKNOWNS USING 2 VARIABLES

One of the classic uses for LP-Problems is in solving shipping problems, where in this course we take a look at a fairly simple setup which allows us to graph a 2-variable Feasible Set. It is recommended the student uses the below described process for configuring the information in such way as to assist in building the correct set of constraints.

Git-tar Corp. makes guitars at two workshops, Memphis, with 90 guitars in stock, and Jackson, with 120 in stock. They have retail stores on Nashville, which needs 60, and Austin, which needs 75. It costs $15 to ship each guitar from Memphis to Nashville, $20 from Memphis to Austin, $24 from Jackson to Nashville, and $18 from Jackson to Austin. How many should be shipped in each of the four routes, and at what minimum total cost?

The diagram at the right indicates the direction of movement.



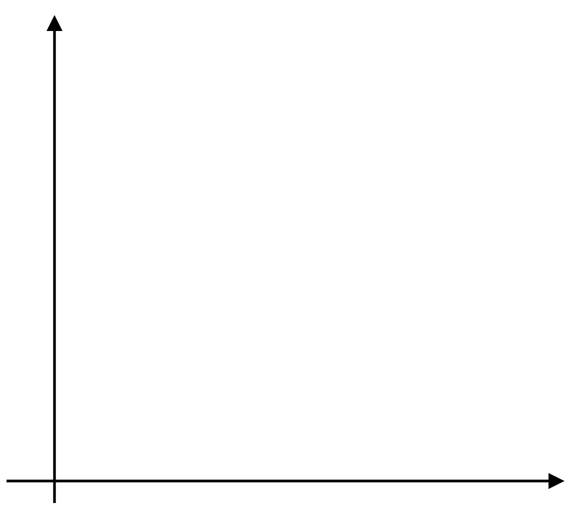
It is best to choose the two quantities being shipped out of a city as *X* and *Y*. Do so for Memphis. Now decide how to represent the quantities shipped out of Jackson based on how many are needed in Nashville and Austin. You **MUST USE** *X* and *Y* to do so.

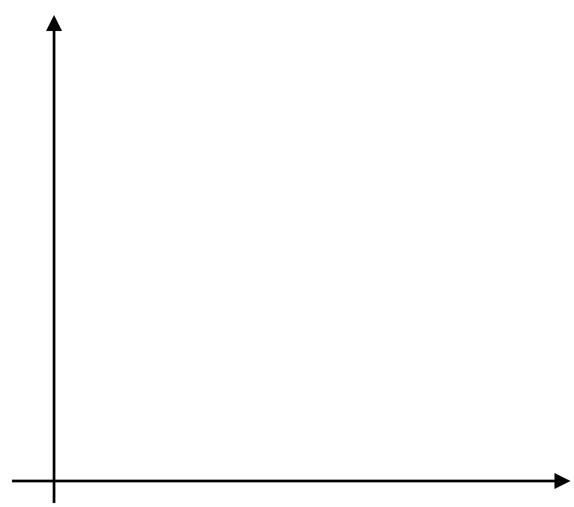
List each quantity next to the city, along with whether it “has” or “needs” that quantity. The clues to setting up constraints lie in these thoughts: if a workshop only has so many on hand, it cannot ship out more than that amount, and each quantity shipped(in the 4 routes) cannot be negative. Simplify, as needed, to put each constraint into preferred form.

Objective Function: to build this, list the shipping cost next to each of the four routes, multiply that cost times each respectively described quantity shipped, and add them all up into one single function. Simplify as much as possible.

EXERCISE: Redo the previous example, but this time select Jackson as the origination point, label the quantities leaving Jackson as *X* and *Y* on a new diamond-shaped diagram. Fill out the rest of the diagram accordingly and then build all necessary constraints, plus a new Objective Function.

You should now have two separate, similar but different, LP-Problems. To see that both actually give the same solution, graph both below, and then detail each of the four routes' quantities while also noticing you should have the same minimum cost.





EXERCISE: A produce company regularly ships apples, bananas, and cherries to Chicago. The truck carries exactly 180 crates and is always full on each trip. Their customers want them ship at least 20 of each type of fruit. The sales manager wants at least as many crates of cherries as apples, and at least twice as many bananas as cherries. The profit is $15 on each crate of apples, $10 on each crate of bananas, and $24 on each crate of cherries. How many crates of each should be sent each shipment to realize the most profit, and what will be that amount of profit?

We have 3 unknowns, in how many crates of each of three types of fruit. In order to model this using only 2 variables, we recall that all three cases of fruit quantities will ALWAYS total 180 crates. So, for simplicity, we will use X = # crates of apples and Y = # crates of bananas. Now, how should we model the # crates of cherries?

SECTION 4.1: BUILDING A SIMPLEX TABLEAU AND PROPER PIVOT SELECTION

$Maximize: 15x + 25y + 18z \quad s.t.$

$2x + 3y + 4z \leq 60$

$4x + 4y + 2z \leq 100$

$8x + 5y \quad \leq 80$

$x \geq 0\ ,\ y \geq 0\ ,\ z \geq 0$

a) Build Equations out of each of the constraints above by introducing Slack Variables
b) Convert the Objective Function to an Equation by setting it = M. Move all other variables to the same side as M, and place below the Constraint/Slack Variable Equations

c) Place the Coefficients from all Equations into a Simplex Tableau, labeled above with variables indicating their respective columns

d) The first step here in choosing a Pivot element is to note the “most negative number” in the bottom row(except possibly the bottom right corner value for M). Which Column is that?

e) Once a Column is selected, calculate the ratios of each Right-Hand side value over the pivot column value. The smallest Non-Negative Ratio indicates the Pivot Row. Which Row has the smallest Non-Negative value?

f) Put the Initial Tableau into your calculator, and save a copy into another matrix
Perform a Pivot on the position Row 1, Column 2 (this is not the Row we chose in part e)
What do you notice has happened in the Right-Hand column? A negative number has appeared, which tells us we are "not feasible", and we have performed an incorrect Pivot. This should never happen when we Pivot from a Feasible situation.

g) Start the Pivot process over, going back to the Initial Tableau in part (c), and Pivot where we decided was the correct Pivot element in part (d) and (e). Write the new Tableau down in the space below.
Notice there are no negative numbers in the Right-Hand column, so we are feasible.

h) List the values of ALL variables, a "new" Current Solution: decision variables, slack variables, and M

i) Determine where the next Pivot should occur, but **DO NOT PIVOT**.

EXERCISE: Given the LP-Problem

$Maximize: 16x + 22y \quad s.t.$

$3x + y \leq 180$

$x + 4y \leq 260$

$x + y \leq 80$

$x \geq 0\ ,\ y \geq 0$

a) Convert the LP-Problem into equations with Slack Variables, also turning the Objective Function into a properly built equation in the space above right

b) Use the equations to build an Initial Simplex Tableau, with appropriate column labels

c) Indicate where you should Pivot in the above Tableau, showing the Ratios used. Then perform the Pivot on your calculator and write out the new Tableau in the space above on the right.

d) Verify that we are Feasible. Then, are we Optimal? Yes or No ? List the "new" Current Solution from this new Tableau

e) Again, indicate where the next Pivot should be done, Ratios shown fully. Perform that Pivot on your calculator and write out the new Tableau below.

f) Feasible? Yes or No Optimal? Yes or No New Current Solution?

SECTION 4.2: USING SIMPLEX TO FIND AN OPTIMAL MAXIMUM SOLUTION

EXERCISE: Furniture Factory produces tables, chairs, and desks. Each table needs 3 hours of carpentry, 1 hour of sanding, 2 hours of staining, with a profit of $12.50. A chair needs 2 hours of carpentry, 4 hours of sanding, 1 hour of staining, with a profit of $20. Desks need 1 hour of carpentry, 2 hours of sanding, 3 hours of staining, with a profit of $16. There are 660 hours of carpentry, 740 hours of sanding, and 853 hours of staining available each week. The manager would like to make as much money as he can.

a) Construct the Linear Programming problem, including an Objective Function and ALL constraints

b) Introduce Slack Variables into each Constraint, and introduce M to the Objective function by turning it into an equation. SHOW ALL of these Equations below

c) Build a fully labeled **Initial** Simplex Tableau above right. Indicate where and why you will pivot.
d) Pivot until you reach an Optimal Solution, and **show the final Tableau**. You may **skip** showing any intermediate Tableaux

e) State your solution in terms of the word problem above

EXERCISE: Given the LP-Problem below, do the tasks that follow.

$Maximize: \ 4x + 6y + 3z \quad s.t.$

$4x + 4y + 8z \leq 800$

$8x + 6y + 4z \leq 1800$

$2x + 4y + 6z \leq 600$

$x \geq 0 \ , \ y \geq 0 \ , \ z \leq 0$

a) Introduce Slack Variables into each Constraint, and introduce M to the Objective function by turning it into an equation. Build a fully labeled **Initial** Simplex Tableau. Indicate where and why you will pivot.

b) Pivot until you reach an Optimal Solution, and **show the 2nd and Final/Optimal Tableau**.

c) State your solution, ALL Variables

EXERCISE: Given the LP-Problem below, perform the tasks which follow

$$\begin{aligned}
&Maximize: 3x+5y+5z \quad s.t.\\
&x+y \quad +z \le 100\\
&3x+2y+4z \le 210\\
&x+2y \qquad \le 150\\
&x \ge 0\ ,\ y \ge 0\ ,\ z \le 0
\end{aligned}$$

a) Introduce Slack Variables into each Constraint, and introduce M to the Objective function by turning it into an equation. Build the Initial Simplex Tableau, and put it into your calculator.

b) Decide where and why you will pivot. Following proper procedures, you should look for "the most negative value" in the bottom row. But here we have a tie, in -5 in both columns *y* and *z*. Which one should we use? It actually should not matter, so we will do it both ways and see what happens.

c) Select the proper Row in Column *y*, then Pivot until you reach an Optimal Solution, and **show the final Tableau**. You may **skip** showing any intermediate Tableaux

d) Go back the Initial Tableau and repeat the process for Column *z*, choosing the correct Pivot Row.

e) Did you get the same Optimal Solution both times? What is it?

SECTION 4.3: NON-STANDARD CONSTRAINTS AND MINIMUM OBJECTIVE FUNCTIONS

NON-FEASIBLE STARTING SOLUTION

$Maximize: 4x + 6y \quad s.t.$

$2x + y \geq 15$

$4x + 5y \leq 100$ a) Rewrite Constraints 1 and 3 to proper form

$x + y \geq 10$

$x \geq 0\ ,\ y \geq 0$

b) Introduce Slack Variables and Construct an initial Simplex Tableau

c) Suppose we choose to look at the Negative Number in the Right Column, Row 1, and then choose Column 2, the Y-column. Calculate non-negative ratios, which indicate a Pivot in Row 3, not our originally noted Row 1. Perform this Pivot and show the new Tableau. Are we Feasible? (no, not yet)

d) Continue Pivoting until reaching an Optimal Solution, SHOW the final tableau
List the value of every variable(X,Y,U,V,W, and M)

e) Go back to the Initial tableau, and instead of Column 2(Y-var), select Column 1(X-var) for your first Pivot. Solve completely and compare Optimal Solution to the first time.

SOLVING A MINIMUM PROBLEM BY CONVERTING TO A MAXIMUM

$Minimize: 18x + 36y \quad subject \ \ to$

$3x + 2y \geq 24$

$5x + 4y \geq 46$

$4x + 9y \geq 60$

$x \geq 0 \ , \ y \geq 0$

a) Convert the Min to Max and fix ALL Constraints

b) Build the Initial Simplex Tableau, indicate where/why you will Pivot and perform the pivot
Show BOTH the Initial and 2nd Tableaux

c) Finish the Pivoting process until Optimal, SHOW the final Tableau

d) List the values of X and Y, found in the bottom row of their respective columns
Remember to Multiply the M-value by -1 for the proper solution to the Minimum

EXERCISES: Solve each of the following LP-Problems. Convert Constraints and/or Objective Functions to necessary form, as needed. Show each Initial Tableau, Pivot choice, and Final Tableau. State each Optimal Solution.

i)
$$
\begin{aligned}
&Minimize: 30x + 15y + 28z \quad s.t. \\
&5x + 3y + 4z \geq 45 \\
&5x + 6y + 8z \geq 120 \\
&10x + 3y + 7z \geq 150 \\
&x \geq 0\ ,\ y \geq 0\ ,\ z \geq 0
\end{aligned}
$$

ii)
$$
\begin{aligned}
&Maximize: 10x + 50y + 30z \quad s.t. \\
&3x + 6y + 2z \leq 450 \\
&5x + 16y + 8z \geq 120 \\
&3x \quad + y \quad + z \geq 300 \\
&x \geq 0\ ,\ y \geq 0\ ,\ z \geq 0
\end{aligned}
$$

SECTION 4.4: SENSITIVITY ANALYSIS

A Linear Programming problem and its Optimal Tableau are given below:

$Maximize: 3x+5y+2z \quad s.t.$
$2x+4y+2z \le 340$
$3x+6y+4z \le 570$
$2x+5y+z \le 300$
$x \ge 0\ ,\ y \ge 0\ ,\ z \ge 0$

x	y	z	u	v	w	M	
0	1	0	$-5/2$	1	1	0	20
0	-1	1	1	0	-1	0	40
1	3	0	$-1/2$	0	1	0	130
0	2	0	$1/2$	0	1	1	470

Sensitivity Analysis takes many different looks at LP-Problems, asking various questions about changes to the original problem and how that would affect the Optimal Solution. A more advanced course might look at several such questions, while in this course we will ask just one of them: what would we do differently if the Right-Side Constant in one of our Constraints changed, possibly up or down?

The idea, however, is to answer this ***without*** redoing Simplex from the start. Instead, we can use information within the final Tableau itself to make the necessary updates in solution values, and then read off that new solution. The key to this is connecting the Constraint whose right-side is being changed, by an amount we will call ***h***, to which Slack Variable gains or loses from that change. The column in the Final Tableau for that Slack Variable then provides us what multiple of ***h*** should be applied to the corresponding Right-Hand Side(RHS) value in the Tableau.

Before we make any changes, list the values of ALL variables:

$x =$, $y =$, $z =$, $u =$, $v =$, $w =$, $M =$

EXAMPLE: Suppose the RHS of Constraint **#1** were to be changed to 346.
i) What is the value of h? (How much did the right-side constant change and what direction?)

ii) Which column's values should you use? (Which Slack Variable is connected to Constraint 1?)

iii) Calculate each of the New RHS values as follows: $Old\ \ RHS\ +\ h(slack\ \text{var}\ column) = New\ \ RHS$

For example, in the top row: $20\ +\ h(-5/2) = ???$ Find this value and those in the remaining rows.

Now that we have new values for the RHS of the Final Tableau(every other value in that Tableau is unaltered), list the "New" Optimal Solution:

$x =$, $y =$, $z =$, $u =$, $v =$, $w =$, $M =$

b) Suppose instead the RHS of Constraint **#1** were to be changed to 330(from the original 340).
i) What is the value of h this time?

ii) Which column's values should you use?(Same Constraint, so….?)

iii) Calculate the New RHS values for the Tableau: $Old\ \ RHS + h(slack\ \text{var}\ column) = New\ \ RHS$

State the new Optimal Solution values.

$x =\qquad , y = \qquad , z = \qquad , u = \qquad , v = \qquad , w = \qquad , M =$

c) If the RHS of Constraint **#3** were to be changed to 330, determine the NEW Right-hand-side values, and then State the new Optimal Solution values.
i) What is the value of h?
ii) Which column's values should you use?
iii) Calculate the New RHS values for the Tableau

State the new Optimal Solution values.

$x =\qquad , y = \qquad , z = \qquad , u = \qquad , v = \qquad , w = \qquad , M =$

Look back at each of the "New" Optimal Solutions we found, each from a different change made to the right-side value one of the Constraints. Each time, the "New RHS" values never contained any Negative values. This means we are still "Feasible", and thus can consider the new solution to be Optimal(remember that the Bottom Row of the Tableau was not changed, which had indicated to us at the outset we had an Optimal Solution originally).

Another question we can explore is now this: How far up or down can we change the right-side value of each Constraint before the new solution is NOT feasible? It can only move so far each way, right?

Once again the key lies in the calculation: $Old\ \ RHS\ +\ h(slack\ \text{var}\ column) = New\ \ RHS$
Focus on that "*New RHS*", which we do not want to be Negative, which would indicate a Non-Feasible situation. That leads to the need for this: $Old\ \ RHS\ +\ h(slack\ \text{var}\ column) \geq 0$
We need to set up such an Inequality for each row in the final Tableau(EXCEPT the bottom Row).

EXAMPLE:If the RHS of Constraint **#1** were changed to $340 + h$, find the range of *h* so that we are still feasible. As with specific values for *h*, it is just this change value we use(the 340 original amount is irrelevant). Also similar is that we identify which Slack Variable we are connected to, here it is again *u*.

So, for example, in Row 1, we would set up $Old\ \ RHS\ +\ h(slack\ \text{var}\ column) \geq 0$ as follows:

$20 + h(-5/2) \geq 0$ Solve for *h*

Now, construct and solve similar Inequalities for Rows 2 and 3(NOT the Bottom Row)

We would like to have a single statement about *h*, in the form $a \leq h \leq b$. If it is not obvious from the three(in this example) separate Inequalities how to consolidate them, graph each one on a separate number line, but rather than just placing an arrow to indicate which direction, shade it all the way to the end. Then decide what portion of ALL 3 number lines is colored in, which represents the necessary $a \leq h \leq b$.

SECTION 4.5: SIMPLEX METHOD AND DUALITY

Earlier in the chapter, we saw a method for solving a "*Minimize*" type of LP-Problem, but here we explore another way to do so. It was discussed in lecture that every LP-Problem has a Primal form and a Dual form. If the Primal is a Max-problem, its Dual is a Min-problem, while if the Primal is a Min-problem, then its Dual is a Max-problem. It really depends on which one we need to solve, which is our "original". That one is then the "Primal", our *primary* solution goal.

In this course, we will keep our focus more on the Primal being a Min-problem, and so its Dual will be a Max-problem. As we know, Simplex only solves Max-problems(by design), which we will utilize as an option for solving Min-problems by solving their Duals instead.

How to build the Dual from our Min-problem? As was shown in class, we can de-construct any LP-problem in "standard Minimum" form into matrices A, B, C, and X , and represent the problem as:

$$\begin{aligned} &Minimize: CX \quad s.t. \\ &AX \geq B \\ &X \geq 0 \end{aligned}$$

Identify all four matrices for the LP-Problem:

$$\begin{aligned} &Minimize: 18x + 36y \quad s.t. \\ &3x + 2y \geq 24 \\ &5x + 4y \geq 46 \\ &4x + 9y \geq 60 \quad x \geq 0 \text{ , } y \geq 0 \end{aligned}$$

$$A = \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix} \qquad B = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \qquad C = \begin{bmatrix} \quad & \quad \end{bmatrix} \qquad X = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Since the Dual has the form

$$\begin{aligned} &Maximize: B^T U \quad s.t. \\ &A^T U \leq C^T \\ &U \geq 0 \end{aligned}$$

, we need the Transposes of *A*, *B*, and *C*. Recall from lecture that the transpose of a matrix with dimensions "m by n" is one of dimensions "n by m" where either each row of the original becomes the corresponding column, or each column of the original becomes the corresponding row of the transpose. Build A^T , B^T , *and* C^T and then use them with a proper choice of *U*(think slack variables) to construct the Dual Max-problem.

$$A^T = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}, \; B^T = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}, \; C^T = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Now build the Initial Simplex Tableau. Recalling it is now a Max-problem, most of our pivoting will be done in “standard” fashion. What did you use for the Slack Variables? Since *u*, *v*, and *w* are now the main variables in this Max-problem, why not use *x* and *y* as Slack Variables? There is actually a very good reason for doing exactly that, which we will get to later.

u	*v*	*w*	*x*	*y*	*M*	
					0	
					0	
					1	0

Pivot in “Standard” fashion until you reach an Feasible Optimal Tableau, and fill in the below Tableau with those values.

u	*v*	*w*	*x*	*y*	*M*	
					0	
					0	
					1	

Now we need to understand what information this final tableau is offering us. We can read the values of the variables *u*, *v*, *w*, *x*, and *y* in the way we always have, but that would be the Solution to the Max-problem we actually solved here. We are not interested(at the moment) in this solution. However, the values of *x* and *y* for the Primal Min-problem are available, where we simply look at the bottom of their respective columns and note the Bottom Row value in each. Should we desire to know the values of the 3 Slack variables, those are available the same way, at the bottom of their respective columns.

What about M? Not accidentally, the value of M which is the Maximum of the Dual we just solved is identical to the value of M at Optimal Solution for the Primal Min-problem we wanted to solve.

For good practice and a convincing demonstration that Duality correctly provides us the Optimal Solution to a Min-problem, here is the Initial Tableau for that Min-problem. Use the methods from Section 4.3 to pivot until you reach the Optimal Solution and compare to the one we read in the Dual solution above.

x	*y*	*u*	*v*	*w*	*M*	
–3	–2	1	0	0	0	–24
–5	–4	0	1	0	0	–46
–4	–9	0	0	1	0	–60
18	36	0	0	0	1	0

Use Duality to solve each of the Min-problems in the exercises below.
i) Convert to Max/Dual form, ii) Build an Initial Tableau, iii) Pivot until Optimal, iv) State Solution

a)
$$\begin{aligned} &Minimize: 2x+5y \quad s.t. \\ &4x+y \geq 40 \\ &2x+y \geq 30 \\ &x+3y \geq 30 \\ &x \geq 0 \ , \ y \geq 0 \end{aligned}$$

b)
$$\begin{aligned} &Minimize: 14x+27y+9z \quad s.t. \\ &7x+9y+4z \geq 60 \\ &10x+3y+6z \geq 80 \\ &4x+2y+z \geq 48 \\ &x \geq 0 \ , \ y \geq 0 \ , \ z \geq 0 \end{aligned}$$

SECTION 8.1 MARKOV PROCESSES

Students at a certain large university in a big Midwestern city use the library in measurable patterns. The day after a student goes to the library, the student will go again the next day 10% of the time, and will not go 90% of the time. On the day after a student did not go to the library, they will again not go 30% of the time, but will go to the library 70% of the time. According to library records, 40% of the students visited the library today.

a) Fill in the Transition Matrix below, using Probability values taken from the %-values described.
For example, students who went to the Library today are in a Current State L, and if they go again the next day, L again, 10% of the time, then 0.10 should be placed in the top left spot of the matrix.

Current State

$$\text{Next State} \quad A = \begin{matrix} L \\ NL \end{matrix} \overset{\begin{matrix} L & NL \end{matrix}}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}$$

Note that when properly filled in, EVERY column of the Transition Matrix will total 1.0, you should always double-check before proceeding. Such a matrix is called a Stochastic Matrix.

b) Now fill in the column matrix X_0 with the probabilities that a random student visited the library today.
Again, we note that the column has a total of 1.0

$$X_0 = \begin{matrix} L \\ NL \end{matrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

We use the general formula $X_{n+1} = AX_n$ to find "the next day's probabilities" from the current day.
This means we can use X_0 above to find X_1. In our formula, n = 0, so we are setting up this:

$X_{0+1} = AX_0$, and so $X_1 = AX_0$ Calculate those probabilities: $X_1 =$

c) What about the next day? Two days from today? We use $X_2 = X_{1+1} = AX_1$

Use your X_1 to find the probabilities in X_2:

d) Let's find another option. We noted that $X_2 = AX_1$, but we also saw that $X_1 = AX_0$, so if we place this equivalent of X_1 into $X_2 = AX_1$ getting the following conclusion: $X_2 = AX_1 = A(AX_0) = A^2 X_0$

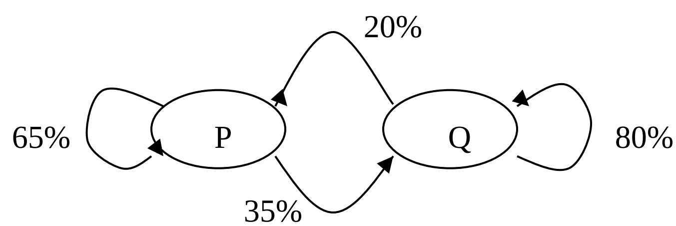
We can calculate X_2 directly from X_0. Use your calculator to do so, and verify it is the same probabilities as we found in part (c).

e) We can extend the idea further, using X_0 to find the probabilities any number of days into the future we wish to find. For example, if we want *n* days into the future, we can use $X_n = A^n X_0$

Use this to find the probabilities 5 days from today. Round to 3 decimal places, as necessary.

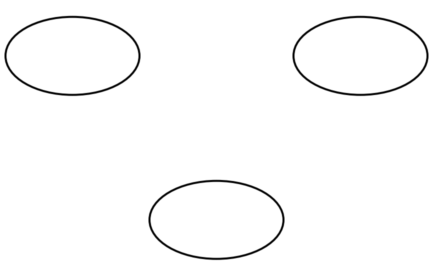
TRANSITION DIAGRAMS can be used to represent movement from one state to another, and for helping set up a proper Transition Matrix.

Build a fully labeled Transition Matrix, A



Now use this Transition Matrix to build a Transition Diagram:

$$A = \begin{array}{c} \\ E \\ F \\ G \end{array} \begin{array}{c} \begin{array}{ccc} E & F & G \end{array} \\ \begin{bmatrix} .6 & .3 & .15 \\ 0 & .5 & .75 \\ .4 & .2 & .1 \end{bmatrix} \end{array}$$



EXERCISE: Members of a local gym usage of the facility have been tracked by management. Members who use the gym for a long workout on a given day have been observed to have a long workout the next day 20% of the time, a short workout 50%, and none at all the rest of the time. For those who had a short workout on a given day, 60% will have a long workout the next day, 25% another short workout, and no workout the rest of the time. For those who did not come to the gym on a given day, half will have a long workout and half a short workout.

i) Build a Transition Diagram to represent the workout patterns of the members

ii) Build a Transition Matrix, fully labeled

iii) Suppose on Tuesday, 38% of the members come in for a long workout, 47% of the members come in for a short workout, and the other 15% do not come in that day. Build a Distribution Matrix to reflect Tuesday's probability distribution.

iv) Set up an appropriate calculation and determine what probability we expect on Wednesday for long, short, and no workouts by the gym members. Answer in %-form, rounded to the nearest 0.1

v) Set up an appropriate calculation and determine what probability we expect on Friday for long, short, and no workouts by the gym members. Answer in %-form, rounded to the nearest 0.1

vi) Set up an appropriate calculation and determine what probability we expect on Sunday for long, short, and no workouts by the gym members. Answer in %-form, rounded to the nearest 0.1

SECTION 8.2: MARKOV PROCESSES AND STEADY STATES

We revisit the university students and their library habits. Recall the Transition Matrix:

$$A = \begin{array}{c} \\ L \\ NL \end{array} \begin{array}{c} \begin{array}{cc} L & NL \end{array} \\ \begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix} \end{array}$$

Also, recall this was called a Stochastic Matrix. Because it also has no 0's in the matrix, we now note that it is a Regular Stochastic Matrix, which is a Stochastic Matrix where at least one power of A, A^n, has ALL non-Zero values. Check each of the following to see if they are Regular, raising them to whole number powers in your calculator, as necessary. Which are regular?

i) $\begin{bmatrix} 0 & .2 \\ 1 & .8 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix}$ iii) $\begin{bmatrix} .3 & 1 & 0 \\ .5 & 0 & 1 \\ .2 & 0 & 0 \end{bmatrix}$ iv) $\begin{bmatrix} 1 & .4 & 0 \\ 0 & .5 & 0 \\ 0 & .1 & 1 \end{bmatrix}$

Regular Stochastic Matrices allow for a "steady" probability occurring "in the long run". When enough transitions have occurred, our formula for finding the next time unit's probabilities, $X_{n+1} = AX_n$, becomes the more general $X = AX$, where the probabilities in X are the same before and after a transition.
Returning to our library example, and using $X = \begin{bmatrix} x \\ y \end{bmatrix}$, set up the matrix equation $X = AX$ and turn it into a system of equations in x and y.

$$\begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \qquad & \qquad \\ \qquad & \qquad \end{bmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix} \quad \Rightarrow$$

Simplify the questions and note that they end up being identical. We then add an extra equation based on the fact that $x + y = 1$ is always true. Now solve the system once this is paired up to your above equation.

Your calculator can be used to verify these "steady probabilities". Calculate: $A^n X$ using a sufficiently large value for *n*.

Now that we have steady state probabilities, let's look at an example of this in action. Suppose the large university has 24000 students and on a day late in a school year, 6000 of those students use the library.

Use your transition matrix A to determine how many of those 6000 will visit/not visit the library the next day.

Next, use A to determine how many of the 18000 who did not use the library will visit/not visit the library the next day.

Add together the numbers of students who will visit the library the next day. Do you get 6000?
Add together the numbers of students who will not visit the library the next day. Do you get 18000?

EXERCISE: Use the process from above(not your calculator) to find the Steady Probabilities the Transition Matrices: a) $\begin{bmatrix} 0.1 & 0.65 \\ 0.9 & 0.35 \end{bmatrix}$ b) $\begin{bmatrix} 1/4 & 2/5 \\ 3/4 & 3/5 \end{bmatrix}$

CALCULATOR EXERCISE: Use your calculator to raise the below Transition Matrix to each listed Power, and then use the →FRAC command to see if Fraction form is available yet

$$A = \begin{bmatrix} 0.3 & 0.6 & 0.4 \\ 0.7 & 0.3 & 0 \\ 0 & 0.1 & 0.6 \end{bmatrix} \qquad \text{i) } A^{10} \qquad \text{ii) } A^{20} \qquad \text{iii) } A^{40} \qquad \text{iv) } A^{100}$$

EXERCISE: We revisit the gym member usage exercise from the Section 8.1 pages. Below is the Transition Matrix we built for this situation. Also given is the Distribution Matrix for Tuesday.

$$A = \begin{array}{c} \\ L \\ S \\ N \end{array}\begin{array}{c} \begin{array}{ccc} L & S & N \end{array} \\ \begin{bmatrix} .2 & .6 & .5 \\ .5 & .25 & .5 \\ .3 & .15 & 0 \end{bmatrix} \end{array} \qquad X_{Tuesday} = \begin{array}{c} L \\ S \\ N \end{array}\begin{bmatrix} .38 \\ .47 \\ .15 \end{bmatrix}$$

i) Show the necessary calculation for the following Tuesday, round-off values to the nearest 0.1%

ii) Find the %'s, rounded to the nearest 0.1, for a Tuesday 4 weeks later. And 8 weeks later.

iii) Now suppose the member usage was 5% long workout, 15% short, and 80% none on a day in winter when the city is hit by a massive blizzard. Build an appropriate Distribution Matrix, and find the %'s for 1 week, 4 weeks, and 8 weeks later. Compare to the results above in part (ii).

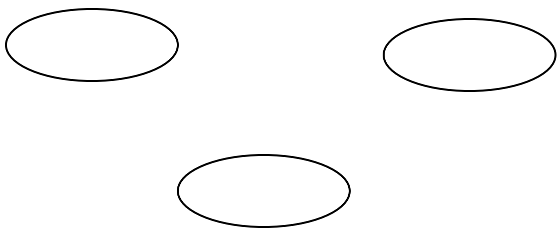
Once we get far enough forward from any random day, the values get pretty similar regardless of the Initial Distribution.

SECTION 8.3: ABSORBING STATES AND ABSORBING MATRICES

Earlier we looked at the concept of a Regular Stochastic Matrix, where it was considered Regular if some Power of the Stochastic Transition Matrix has ALL Non-Zero values. Raise each of the below Transition Matrices to the 20^{th}, 50^{th}, and 100^{th} powers.

$$A_1 = \begin{bmatrix} 0 & .2 & 0 \\ 0 & .3 & .4 \\ 1 & .5 & .6 \end{bmatrix} \qquad A_2 = \begin{matrix} & \begin{matrix} E & F & G \end{matrix} \\ \begin{matrix} E \\ F \\ G \end{matrix} & \begin{bmatrix} .5 & 0 & .4 \\ .2 & 1 & 0 \\ .3 & 0 & .6 \end{bmatrix} \end{matrix}$$

What do you see happening? A_1 is definitely Regular since at all three exponent levels, it has no Zeros, while A_2 never loses all of its Zeros. Therefore A_2 is not Regular. So, what is it? Build a Transition Diagram for Matrix A_2 using the bubbles below.



Do you see that state F never sends anything in the directions of either E or G? In fact, all it does is bring in from those states along with keeping hold of everything it already has. State F "Absorbs" anything that comes its way. Such states are accordingly called Absorbing States.

How do we identify Absorbing States? In the Transition Matrix, it is quite easy. Do you notice that State F has a 1 on the main diagonal of the matrix(from top left to bottom right)? Notice this means that 100% of what is currently in F this time will still be in F next time, nothing gets out, and has been "absorbed" by F. In your Diagram, F should have no arrows pointing towards anything but itself, again indicating that everything currently in F will again belong to F next time, all 100%.

So, what does this make matrix A_2? We would like to know if it is what we will now call an Absorbing Stochastic Matrix. It is Stochastic, but is it Absorbing? For this to be the proper description, it needs:
a) at least one Absorbing State. (Yes, we have already verified F is Absorbing)
b) ALL non-Absorbing states must have a pathway, directly or indirectly, to at least one Absorbing state.

Do E and G have pathways to F? The Diagram is perhaps easier to use in deciding this. Look at E in your Diagram. Notice it certainly has an arrow directly to F. But what about G? It does not have a direct pathway into F, no arrow in the Diagram from G to F. However, G does have an arrow to E, and as we noted, E has an arrow to F. This is an "indirect" pathway to F from G.

Conclusion: is Matrix A_2 an Absorbing Stochastic matrix?

EXERCISE: decide whether each of the following is Absorbing or not.

i) $\begin{array}{c} \\ U \\ V \\ W \\ Y \end{array} \begin{array}{c} \begin{array}{cccc} U & V & W & Y \end{array} \\ \begin{bmatrix} .2 & 0 & 0 & 0 \\ .1 & 0 & 1 & .5 \\ .3 & 1 & 0 & 0 \\ .4 & 0 & 0 & .5 \end{bmatrix} \end{array}$

ii) $\begin{array}{c} \\ J \\ K \\ L \\ M \end{array} \begin{array}{c} \begin{array}{cccc} J & K & L & M \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & .1 \\ .3 & 0 & 0 & .5 \\ .3 & 0 & 1 & 0 \\ .4 & 0 & 0 & .4 \end{bmatrix} \end{array}$

Now we need to use an Absorbing Matrix to see what it tells us.

STEP 1: reorder the listings of the states across the top, and down the side, such that we list all of the absorbing states first, then the non-absorbing states. Call this form of the matrix the Standard Form. You may wish to use a Diagram to help the rewriting process.

$\begin{array}{c} \\ J \\ K \\ L \\ M \end{array} \begin{array}{c} \begin{array}{cccc} J & K & L & M \end{array} \\ \begin{bmatrix} .3 & 0 & .2 & 0 \\ .3 & 1 & .4 & 0 \\ 0 & 0 & .3 & 0 \\ .4 & 0 & .1 & 1 \end{bmatrix} \end{array}$

$\begin{array}{c} \\ K \\ M \\ J \\ L \end{array} \begin{array}{c} \begin{array}{cccc} K & M & J & L \end{array} \\ \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} \end{array}$

$\left[\begin{array}{c|c} I & S \\ \hline 0 & R \end{array}\right]$

Once you have reordered the matrix(one option of the list of states is given, others are possible), place lines in the matrix to help determine Matrices S and R, as shown in the Standard matrix setup above right.

STEP 2: Calculate the following: $S*(I-R)^{-1}$ using your calculator and place it back into the spot where S was in the Standard form matrix, and now we have the Stable matrix. It should look like this:

$$\left[\begin{array}{c|c} I & S*(I-R)^{-1} \\ \hline 0 & 0 \end{array}\right]$$

It's important to note which states are listed above $S*(I-R)^{-1}$, which are the Non-Absorbing states, as well as those along the side, which are the Absorbing states. The probabilities in $S*(I-R)^{-1}$ indicate the probabilities or proportions of values starting in each Non-Absorbing state that will eventually be absorbed into each of the Absorbing states.

Unlike when we looked at Regular matrices, now the amount starting in each of the Non-Absorbing states matters very much in determining what will eventually end up in the absorbing states.

STEP 3: Calculate $(I-R)^{-1}$, which is called the Fundamental matrix. Note that we still use the same Non-Absorbing state listing on top. Add the values in each column, where the total represents the expected number(or average) of transitions an element starting in that state will circulate until landing in one of the Absorbing states. It is not suggesting which Absorbing state.

EXERCISE: a 2-year college has been looking at the patterns of progress their students make towards completion, or not, of their degree. They find that for incoming freshman students, 30% do not take and/or pass enough credit hours to become sophomores, and so are freshmen the next year, 60% do pass enough and become sophomores, and 10% drop out and never return to school. Among those starting the year as sophomores, 50% earn enough credits to graduate, 35% do not earn enough to graduate, and so return the next year as sophomores again, and the other 15% drop out and never return.

STEP 1: Fill in the Transition Matrix. Then reorder it into Standard Form(use a Diagram if needed)

$$\begin{array}{c} \\ F \\ S \\ D \\ G \end{array} \begin{array}{c} \begin{array}{cccc} F & S & D & G \end{array} \\ \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array} \qquad\qquad \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

STEP 2: Identify Matrices S and R. Use your calculator to find $S*(I-R)^{-1}$, and place it into the proper upper right section of the now Stable Matrix below. Properly label the matrix.

$$\left[\begin{array}{c|c} I & \quad \\ \hline 0 & 0 \end{array}\right]$$

Suppose there are 5000 freshmen and 4000 sophomores at the start of this school year. Use the Stable Matrix probabilities to determine how many of the 9000 students total will eventually graduate and how many will eventually drop out(round to the nearest whole number, as needed).

$\#Graduate = \#F * \Pr(G) \ + \ \#S * \Pr(G) =$

$\#Dropout = \#F * \Pr(D) \ + \ \#S * \Pr(D) =$

STEP 3: Now calculate $(I-R)^{-1}$ using your calculator. Add up each column to determine the average number of years on average for Freshmen and Sophomores to spend at the school before eventually either graduating or dropping out.

EXERCISE: A consumer electronics company always has products in development. They have found that among products less than 1-year into development, 30% go to market, 60% get another year of funding, and 10% are abandoned. For projects over 1-year into development, they have found that 50% go to market, 10% get another year, and 40% abandoned.

a) Complete the labeling of the matrix below and fill in appropriate probabilities for this project cycle. Then convert the Matrix into Standard form.

$$<1 \quad M \quad 1+ \quad Ab$$

b) Now identify matrices S and R. Use S and R to calculate $S*(I-R)^{-1}$, and place it into Stable Matrix form, properly labeled.

c) Suppose the company has 60 projects under 1-year in development and 90 projects over 1-year in development. Use your stable matrix to determine how many are expected to reach market and how many will eventually be abandoned.

d) Use R to calculate the Fundamental Matrix, $(I-R)^{-1}$. Use it to determine the expected number of years each of "less than 1-year" and "over 1-year" projects will spend in development before eventually either being brought to market or abandoned.

SECTION 9.1 STRICTLY DETERMINED GAMES

We open the discussion of Game Theory first by noting we want only to look at aspects of what are called "Zero Sum Games". In these games, the Zero Sum comes from this notion: whatever money one player wins is exactly the amount of money the other player loses. Simply put, it is an exchange of money from the losing player to the winning player, there are no third parties taking a piece of the pie.

For simplicity, we will think of our 2 players as R(for Rows) and C(for Columns), where each players possible outcomes from the Game are represented in a matrix. For example, we have the below Payoff Matrix:

$$\begin{array}{c} \\ R \end{array} \overset{\textstyle C}{\begin{bmatrix} 2 & -1 & 3 \\ -3 & -2 & 4 \end{bmatrix}}$$

Player R looks at the Rows to see their possible payouts. If R chooses to play the option in Row 1, for example, the payoff will be a gain of \$2, a loss of \$1, or a gain of \$3. If R plays Row 2, payoffs possible are a loss of \$3, a loss of \$2, or a gain of \$4.

On the other hand, Player C looks at the Columns to see their payoffs, but must take the OPPOSITE of each number to give their gain/loss. For example, in Column 1, C sees a 2 and a -3. But this means a LOSS of \$2 or a GAIN of \$3. Remember, we said that R would possibly see a gain of \$2 in choosing to play Row 1, and that \$2 must come from player C, so a loss for C.

Let us suppose R sees the 4 in Row 2 and decides to play Row 2 over and over, hoping to hit that \$4 gain, while C notices the -3(so a gain of \$3 for C) in Column 1 and decides to play Column 1 over and over. As things stand, C will win \$3 from R each play unless R changes things up. Which R logically would do, switching to Row 1, since R would win \$2 from C if C keeps playing Column 1. However, now C will realize that repeated plays of Row 1 are occurring, so should choose to move from Column 1 to Column 2, where C now wins \$1 each play.

We are now seeing R play Row 1 repeatedly, and C play Column 2 repeatedly. Does either of them improve their situation by making another change of strategy while their opponent does not change? If you think about it, the answer is "No", neither of them will switch to another option because to do so will hurt their current gain/loss position.

When such a situation occurs, it is called a Strictly Determined Game. The specific gain/loss figure their respective choice targets is called the Value, v, of the game. Here, we have that v = -1, a loss of \$1 for R and so a gain of \$1 for C every time the game is played.

The position in the Payoff Matrix is referred to as a Saddle Point, a value which is both the Minimum of its Row and the Maximum of its Column. Verify in our Payoff Matrix that this is the case in Row 1, Column 2, that the -1 is a Saddle Point in this sense.

Now, we discuss how to identify whether a Payoff Matrix has a Saddle Point or not, and so whether the Game is Strictly Determined or not.

One fairly simple way to do this is to look across each Row and Circle the lowest(minimum) number, and then look at each Column and Box the highest(maximum) number in each.

If you have both Circled and Boxed a specific number, it must have been both a Minimum of its Row as well as a Maximum of its Column, the exact description we just said makes it a Saddle Point.

Try it on our Payoff Matrix:

$$\begin{array}{cc} & C \\ R & \begin{bmatrix} 2 & -1 & 3 \\ -3 & -2 & 4 \end{bmatrix} \end{array}$$

Now try it on the following Payoff Matrices, and decide whether each is Strictly Determined or not. And if it is Strictly Determined, what is its Value, v?

i) $\begin{array}{cc} & C \\ R & \begin{bmatrix} -2 & -1 & 3 \\ 3 & -2 & 0 \end{bmatrix} \end{array}$

ii) $\begin{array}{cc} & C \\ R & \begin{bmatrix} 2 & 8 \\ 5 & 6 \end{bmatrix} \end{array}$

iii) $\begin{array}{cc} & C \\ R & \begin{bmatrix} 3 & -1 & 4 \\ -2 & -2 & 2 \\ 0 & -3 & 5 \end{bmatrix} \end{array}$

APPLICATION: Rick and Chuck are playing a game. Rick has cards with the numbers 2, 5, and 6, while Chuck has cards with the numbers 3, 4, and 9. They must each show a card of their own choosing. If they show cards where one is Even and one is Odd, Rick wins the difference of card values, but if they show cards that are both Even or both Odd, Chuck wins the value of the larger card showing.

a) Construct a labeled Payoff Matrix to reflect the gains/losses each player will incur
b) Determine whether the Payoff Matrix is Strictly Determined. If so, what is the Value?

APPLICATION: Rhonda and Chrissy are competing drive-share operators. Both of them use *Över* and *Ryze* phone apps to locate people seeking rides around town. When both are using *Över*, Rhonda takes \$100 in commissions from Chrissy, while if both use *Ryze*, Chrissy takes \$250 from Rhonda. If they are using different apps, Rhonda using *Över* takes \$150 from Chrissy, while if Chrissy uses *Över*, she takes \$75 from Rhonda.

a) Construct a labeled Payoff Matrix to reflect the gains/losses each player will incur
b) Determine whether the Payoff Matrix is Strictly Determined. If so, what is the Value?

SECTION 9.2 MIXED STRATEGIES

The Strictly Determined Games in 9.1 leave no doubt how the games outcomes will occur after repeated plays of the game force the players into the Saddle Point. So, what might the Value, v, of a game be when it is not Strictly Determined? This will depend on how often each player chooses to play their respective Row/Column options.

We will continue using matrices to find our answer. Thus, let us put the probabilities of R playing each Row into a Row Matrix, $R=\begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}$, and the probabilities of C playing each Column into a Column Matrix, $C=\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Of course, how many elements we have in each of the matrices R and C depends on how many options each has in the Payoff Matrix, which we will call A.

We now can use a very simple Matrix Multiplication, $R \cdot A \cdot C$, the result of which is a 1x1 matrix containing the Value, v, of this particular situation. Understand that v is most likely not an element in the Payoff matrix, but a number that would be the "average" amount won/lost by each player over many plays of the game using the assigned probabilities. If v is positive, R wins on average, and if v is negative, C wins on average.

Given $A=\begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix}$, first analyze it to verify that it is NOT Strictly Determined.

i) Let us assign probabilities for each player. Start with R playing Row 1 70% of the time, and C playing Column 1 40% of the time. Construct both matrices R and C, and the perform the calculation $R \cdot A \cdot C$ What do you get for *v*? Which player is winning on an average play right now?

Since we see a positive value of *v*, R is currently winning, and so we expect C might alter strategy.

ii) Keeping R's probabilities the same, change C to playing Column 1 20% of the time instead of 40%. What is the new value of *v*? Did C improve their situation?

iii) Because that change in strategy did not work for C, perhaps C flips probabilities around, now deciding to play Column 1 80% of the time. What is the new value of *v*? Did C improve their situation?

iv) Now that C seems to be winning on average, they might be content to continue their current % of plays on Columns 1 & 2. But now R might make a change, right? So, select some new values for R and see if the new choices improve the value of *v* in R's favor. List each row matrix R and the *v* it gives.

Probability plays are chosen by each side of the game, but the players will then attempt to improve their situation by adjusting their probabilities. If it improves the value, v, of the game in their favor, perhaps they hold steady, but their opponent is likely adjusting their play. Thus, what happens is both sides will continually try improving their outcome until perhaps we might find that neither can make any moves that will improve their situation.

To explore this, let us go back to the last probabilities for C, which was 80% play of Column 1. If you push R's plays as far as possible towards Row 2, in fact all the way to 100%, R does push v to a value of 0. It seems this is the best R can force to happen for now. But, let us have C make one more adjustment, to 70% in Column 1. So, we have now these matrices and value of v:

$$R = \begin{bmatrix} r_1 & r_2 \end{bmatrix} \quad A = \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix} \quad C = \begin{bmatrix} .7 \\ .3 \end{bmatrix} \qquad v = \begin{bmatrix} -0.5 \end{bmatrix}$$

You should have had $R = \begin{bmatrix} 0 & 1 \end{bmatrix}$ as the last used plays by R. Try changing R's probabilities to any valid choice you'd like. Does it improve R's position? Try again. And again. And again.

It turns out there is nothing R can do anymore, is there? C has found their best, Optimal play probabilities, and no matter what R does, the value of the game will be $0.50 in C's favor. We will explore this further in section 9.3 and find out using Linear Programming and Simplex how to determine what probabilities BOTH players can locate as their optimal strategies.

EXERCISE: We return to the game between Rick and Chuck, where the below matrix represents the payoffs for this game, labeled according to the cards each can choose to play.

$$\begin{array}{cc} & C \\ & \begin{array}{ccc} 3 & 4 & 9 \end{array} \\ \begin{array}{r} 2 \\ R\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & -4 & 7 \\ -5 & 1 & -9 \\ 3 & -6 & 3 \end{bmatrix} \end{array}$$

Choose starting probabilities for each side, calculate v, and determine who is currently winning on average. Then make adjustments to the probabilities for whichever player seems to be losing, to improve their average gain or loss in playing the game.

Are you able to find Optimal Strategy probabilities for either player? It is not too easy, is it?

SECTION 9.3 OPTIMAL STRATEGIES

Now we take a look at how do determine the Optimal Strategy for both players, R and C. A full description of the mathematical building of the following model was shown in lecture and can be read in your textbook. This worksheet presents the necessary steps for solving that mathematical problem.

Let us start with the Payoff Matrix below, which you should verify has no Saddle Point. This means that the Optimal Strategies for our players R and C will not be 100% for either of their possible plays. The first task will be to make all of the elements of the matrix positive numbers, and we prefer adding the smallest necessary Integer in accomplishing this. What would you add to each element of A below?

$$A = \begin{bmatrix} -6 & 7 \\ 3 & -2 \end{bmatrix}$$ Your altered Matrix: $$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Now that we have **ALL Positive** numbers in our matrix, we can proceed. It is far easier to solve this problem from the perspective of Player C, who needs to solve a Maximum Linear Programming problem, as opposed to R needing to solve a Minimum.

First, let us use C's generic probabilities: $c_1, c_2, ..., c_n$ for as many Columns in A as necessary. Multiply the Payoff matrix A times column C with the probabilities, where each row was shown to be $\leq v$ in the full description of the solution. For example, you might get $6c_1 + 2c_2 \leq v$. Do so for our example.

Next, we will divide both sides of each Inequality by *v*, which is a positive value causing no changes to the inequality symbols. Continuing the example made above:

$6c_1 + 2c_2 \leq v$ becomes this: $6\frac{c_1}{v} + 2\frac{c_2}{v} \leq \frac{v}{v}$. Now, we introduce $z_i = \frac{c_i}{v}$, for each *i* from 1 to n. So, $6\frac{c_1}{v} + 2\frac{c_2}{v} \leq \frac{v}{v}$ becomes $6z_1 + 2z_2 \leq 1$, which is a properly built Inequality constraint for use in Simplex. Convert your Inequalities in $c_i's \ to \ z_i's$. Conveniently, our Objective function is to Maximize $z_1 + z_2$. Build the whole LP problem, which should resemble(but IS NOT) the one below:

$Maximize: \ z_1 + z_2$

$3z_1 + 8z_2 \leq 1$

$2z_1 + 4z_2 \leq 1$

$z_1 \geq 0, z_2 \geq 0$

Now, let's solve the LP problem, where it is best to use y_i's as the Slack Variables because in a manner similar to building $z_i = \frac{c_i}{v}$, we also saw in the full solution that we build $y_i = \frac{r_i}{v}$, where of course the r_i's are the probabilities Player R uses as their Optimal play probabilities. Fill in the below Initial Simplex Tableau:

$$\left[\begin{array}{ccccc|c} z_1 & z_2 & y_1 & y_2 & M & \\ \hline & & 1 & 0 & 0 & 1 \\ & & 0 & 1 & 0 & 1 \\ \hline & & 0 & 0 & 1 & 0 \end{array}\right]$$

Choose a Pivot Element, either column 1 or 2 is available, and continue pivoting until you reach the Optimal Solution to the problem. Write out your Final Tableau:

$$\left[\begin{array}{ccccc|c} z_1 & z_2 & y_1 & y_2 & M & \\ \hline & & & & 0 & \\ & & & & 0 & \\ \hline & & & & 1 & \end{array}\right]$$

We want to read off the values of the z_i's in the "normal way" and the y_i's in the "dual way". We also will need *v*. Let us remember we Maximized $z_1 + z_2$ which equals 1/*v*, and so *v* will always be found as the reciprocal of our optimal value of M in the final tableau.

Since $z_i = \frac{c_i}{v}$ for each *i*, we rearrange to get $c_i = v \cdot z_i$. Find, in fractions, each of our c_i's

In similar fashion, we noted that $y_i = \frac{r_i}{v}$, so now $r_i = v \cdot y_i$. Find, in fractions, each of our r_i's

One last item to address, which is that the *v* we found is for the adjusted matrix. To fix this, we reverse the very first step we made, where we chose to add an integer to each Payoff matrix element, and we now subtract that from the current *v* to get the correct *v* for the original Payoff Matrix.

You should have these values: $r_1 = 5/18$ $r_2 = 13/18$, $c_1 = 1/2$, $c_2 = 1/2$, $v = 1/2$

EXERCISES: find the Optimal plays for each player, R and C, in the below Payoff Matrices

i) $A = \begin{bmatrix} -4 & 1 \\ 3 & 2 \end{bmatrix}$ ii) $A = \begin{bmatrix} 2 & -4 \\ -1 & 6 \\ 3 & -5 \end{bmatrix}$ iii) $A = \begin{bmatrix} -2 & 1 & -1 \\ 0 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

What do you notice about the Optimal Play probabilities in part (i)? What was the value, v? Go back and look at the original matrix, and check it for a saddle point.

APPLICATION: the rulers of two small countries, Rey and Caesar, need to decide each year which industry to put research funding into. When both rulers put funding into their energy sectors, Caesar's country will see a gain of $3,000,000 in trade between the two countries, while if both countries put funding into their technology sectors, Rey's country sees a gain of $4,000,000 in trade. If they put funding into different sectors, Rey's country gains $7,000,000 when he funds energy, while Caesar's country gains $5,000,000 when he funds the energy sector.

i) Build a Payoff matrix for this situation. Determine that it is NOT strictly determined, it has no saddle point.

ii) We would like to find the optimal strategies for the two leaders, but notice that using values like 3,000,000 or 7,000,000 are quite unwieldy, especially when we want to make all of the values in our Payoff Matrix positive. Adding 5,000,001 to each would result in truly imbalanced values for the purpose of solving a Simplex Tableau.

So, rebuild the Payoff matrix using numbers in Millions, for example using 3 for 3,000,000. We will just want to remember to think of our ultimate value v as also "in Millions".

iii) Use this new version of the Payoff Matrix to find the Optimal Strategies for each country's leader and decide which of their countries will have an expected gain in trade "on average" each year.