SECTION 5.1: ANTI-DERIVATIVES

We explored taking the Derivative of functions in the previous chapters, but now want to look at the reverse process, which the student should understand is sometimes more complicated.

In a sense, we are asking a question like this:

If $f'(x) = 2x$, then what function, $f(x)$, is it the Derivative of?

Hopefully, this seems fairly obvious that $f(x) = x^2$. BUT, what about $f(x) = x^2 + 7$ or $f(x) = x^2 - 5$ or many other possibilities with a different constant after the x^2 term? There are truly an infinite number of possible functions, all of which have $f'(x) = 2x$. So, to show that we may(or may not) have some constant in the Anti-Derivative function, we use a generic constant, C, to represent that possibility.

Therefore, we would say that if $f'(x) = 2x$, then $f(x) = x^2 + C$. We will look at how to determine C, when enough information is provided, later.

NOTATION: We have a special symbol to use when we wish to find the anti-derivative of a function, called an Integral Symbol, $\int$, which you can think of as telling you to find the anti-derivative of whatever function is in the Integral. Also in the Integral will be a differential, often dx, which is an indicator of what Variable is used in the function. So, for our opening example, you might see this:

$$\int 2x dx$$

This expression is asking you to find the Anti-Derivative of $2x$, and is indicating that x is the variable. Worked out, you would indicate that $\int 2x dx = x^2 + C$

PROPERTIES TO FIND ANTI-DERIVATIVES

Perhaps the most frequently used property will be the Power Rule, which works in reverse fashion the way our Power Rule for Derivatives worked. It says: $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$. In words, it says to Add 1 to the "old exponent" and then Divide by(or Multiply by the Reciprocal of) the "new exponent".

EXAMPLE: $\int x^4 dx = \frac{1}{4+1} x^{4+1} + C = \frac{1}{5} x^5 + C$ Try this on the following exercises:

i) $\int x^8 dx$ ii) $\int x^{-6} dx$ iii) $\int \frac{1}{x^3} dx$ iv) $\int \sqrt{x} dx$

PROPERTIES CONTINUED:

$\int k \cdot f(x)dx = k\int f(x)dx$ allows us to focus on the anti-derivative of $f(x)$ and then multiply by k later.

$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$ allows us to each function(or term) one-at-a-time. There is no need to try doing all items at once if we do not have to do so.

$\int kdx = kx + C$. This is just a reminder that a constant, k, does have an anti-derivative.

$\int e^x dx = e^x + C$. Just as you needed to memorize the Derivative of e^x, also memorize its anti-derivative.

$\int \frac{1}{x}dx = \ln|x| + C$. Another memorization situation. You have hopefully linked *ln x* to its Derivative of $\frac{1}{x}$ and now need to remember the reverse situation, in that when faced with $\frac{1}{x}$, its anti-derivative is $\ln|x| + C$. And, YES, the Absolute Value symbol is important, just in case any negative numbers are involved, which cannot be used with the Natural Log function.

Practice use of these properties, mixing some together as needed:

i) $\int 12x^3 dx$

ii) $\int (6x^2 - 10x + 3)dx$

iii) $\int 6\sqrt[3]{x}dx$

iv) $\int \left(\frac{4}{x^3} + \frac{5}{x^2}\right)dx$

v) $\int \frac{5}{x}dx$

vi) $\int (e^x + x^2)dx$

WHAT ABOUT *C*? We noted that there may, or may not, have been some constant in the anti-derivative function. If there was, it of course "disappeared" when the derivative was taken. But since we are now moving in the other direction, we sometimes have more information that will help us determine specifically what that constant was.

Suppose we are told that $f'(x) = 8x - 5$ and that $f(2) = 11$. This is enough information to determine a specific function *f(x)*, one that has a known constant value instead of *C*. We start, though, by finding the "general anti-derivative", answering the Integral: $f(x) = \int (8x-5)\,dx = 4x^2 - 5x + C$

This is not yet our final answer, though. In order to determine the specific *C*, we now use the second piece of information, $f(2) = 11$. It tells us that when 2 is plugged into *f(x)*, the output should be 11. We solve this equation, then, for *C*:

$$11 = 4(2)^2 - 5(2) + C \qquad \text{which says} \quad C = 5$$

Now we have a very specific function, $f(x) = 4x^2 - 5x + 5$ that satisfies both of the conditions.

EXERCISE: using the same $f'(x) = 8x - 5$, find specific functions *f(x)* for each of these conditions:

i) $f(4) = -13$

ii) $f(-3) = 6$

EXERCISE: Find the specific function *f(x)* which satisfies: $f'(x) = 9x^2 + 3x \;\; and \;\; f(1) = 5$

EXERCISE: Find the specific function *f(x)* which satisfies: $f'(x) = 3\sqrt{x} + 6x \;\; and \;\; f(4) = 1$

SECTION 5.2: INTEGRATION BY SUBSTITUTION

Sometimes the function whose Anti-Derivative we seek is not among our simpler cases where these 3 properties explored in 5.1 were used: $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$, $\int e^x dx = e^x + C$, or $\int \frac{1}{x} dx = \ln|x| + C$.

Instead, we seem to be looking at the result of a Chain Rule derivative, and so we now present a process for "undoing" a Chain Rule style derivative. The 3 properties mentioned above will be essential, since we ultimately model our given function to fit one of these.

For example, suppose we have this Integral: $\int (2x+5)^7 \, dx$. Looking at it, and thinking derivatives, one might expect that $(2x+5)^8$ would make sense. However, it is not quite that simple, because we need to address a possible coefficient(and other details as we look at more complicated examples).

STEP 1: Choose the "inside" function(as we did learning Chain Rule), and call it *u*, here $u = 2x+5$

STEP 2: Take the Derivative of *u*, but use differential notation: $\frac{du}{dx} = 2$ and multiply by *dx*. This can be done in one step, giving us this: $du = 2dx$

STEP 3: The Substitution Step, where here we look at one of several ways it can be handled. Think "Remove and Replace", where you identify a piece of the original Integral for removal and replace it with an equivalent piece in your substitution setups.

Start with removing $2x+5$, which we have said is equivalent to u, so $(2x+5)^7$ becomes u^7

Next, we see only dx still in the original Integral, but our setup has $du = 2dx$. To get dx by itself, multiply both sides by ½, and we now have this: $\frac{1}{2}du = dx$. Remove *dx*, replace with $\frac{1}{2}du$. Our new Integral is this: $\frac{1}{2}\int u^7 du$, which is now a relatively simple anti-derivative: $\frac{1}{2} \cdot \frac{u^8}{8} + C$

STEP 4: Return this anti-derivative back to the original variable, reversing use of $u = 2x+5$, where we now have $\frac{1}{2} \cdot \frac{u^8}{8} + C = \frac{1}{16}(2x+5)^8 + C$

Good practice would be to use Chain Rule on this function and verify that its derivative is the function in our original Integral.

EXERCISES: Use the Substitution process to find each Integral

a) $\int 8x\left(x^2+3\right)^5 dx$

$u =$

$du =$

b) $\int x^2 e^{x^3-5} dx$

$u =$

$du =$

c) $\int \frac{6}{2x+7} dx$

$u =$

$du =$

MORE ADVANCED EXERCISES:

i) $\int 7x\left(3x^2+1\right)^9\,dx$

$u =$

$du =$

ii) $\int 8x\sqrt{3x^2+2}\,dx$

$u =$

$du =$

iii) $\int \frac{6x^2+8}{\left(x^3+4x\right)^6}\,dx$

$u =$

$du =$