SECTION 3.1: EXPONENTIAL GROWTH AND CONTINUOUS COMPOUNDING

One of the most frequent uses of the Exponential Function, $f(x) = e^x$, in Business Applications is the formula that models Continuous Compounding of Interest. It also models Constant Growth of, for example, the value of assets like a house or artwork. You should have encountered this formula:

$A = Pe^{rt}$ and perhaps also its counterpart, called Present Value: $P = Ae^{-rt}$

First, a use of $A = Pe^{rt}$, where P = amount invested, r = annual interest rate, and t = time in years

Example: Bob invests $8000 at 4.5% compounded continuously(c.c. from now on) for 6 years. What will it be worth? Just plug in each value into its appropriate place in the formula. Make sure to correctly use your calculator, and ask a TA or your Instructor(or a tutor) for help if you are not sure.
On a TI-83/84, the command should look like this: $8000e^\wedge(.045*6)$, and you get $10479.72
DO NOT use this form: $8000e^\wedge(.045)(6) = 50209.34$

Exercises: find each of the following amounts

i) $6400 at 7.2% c.c. for 13 years

ii) $25000 at 3.75% for 30 years

iii) $13250 at 5% for 4 years

iv) $8000 for 16 years at 6.18%

Present Value: to find the "present value" of a future targeted amount, you can use either formula, so long as the "future amount" is plugged in for *A*.

What is the present value of $10000 at 6% for 8 years? Either calculate $P = 10000e^\wedge(-.06*8)$ or set up the following: $10000 = Pe^\wedge(.06*8)$ and solve for *P*. You should get $6187.83 in both cases.

i) How much must be invested now to be worth $24000 in 7 years at 5.5%?

ii) How much must be invested now to be worth $7000 in 15 years at 4.85%?

HOW LONG? This is another often asked question, which can sometimes be as simple as "How long to double an investment?" or maybe more specific, "How long for $4000 to become $15000?". They both are really just different versions of the same process we need to solve. When asked "How long to double", one can simply pick an amount, say $1000 doubling to $2000, or even simpler, $1 to $2. The growth rate of an investment is the same regardless of the amount of money.

Let's determine how long it takes for $3000 to become $10000 at 6% c.c. Based on the description of the situation, decide what values are *A* and *P*, respectively. Which is "present" and which is "future"? Once decided, place them in the appropriate spots to our formula, and of course the interest rate. Do you get:

$$10000 = 3000e^{0.06t}$$

First, divide away the 3000 on the right side, which we must do on both sides according to our rules of solving equations from Algebra. 10000/3000 simplifies to what value? Properly done, it should be 10/3, **NOT** 3.33

You should now have: $10/3 = e^{0.06t}$

Use Logarithm properties to convert into Ln-form: $\ln(10/3) = 0.06t$, at which point we can now simply divide both sides by 0.06. What do you get for *t*?

Solve each of these situations:

i) How long does it take to double an investment at 4.8% c.c.?

ii) How long does it take for an investment of $7000 to grow to $35000 at 6.3% c.c.?

iii) Mike and Carole are saving for a down-payment on a house. If they put away $18000 today at 5.75% c.c., how long until that is worth the $60000 they need for their dream house?

SECTION 3.2: DERIVATIVES OF THE EXPONENTIAL AND NATURAL LOGARITHM

Use of the Definition of the Derivative is a rather complicated way to establish the derivatives of two related functions that come in quite handy for Business calculus work, specifically the below functions:

$$f(x) = e^x \qquad \textit{and} \qquad f(x) = \ln x$$

Fortunately, their derivatives are relatively simple, but this comes with a warning: YOU MUST MEMORIZE what each one's derivative looks like. There will be no "back-up plan" for getting the derivative of each when necessary, you just have to remember them. So, as was presented in lecture, here are the derivatives:

$\dfrac{d\left[e^x\right]}{dx} = e^x$, where very simply, the derivative of e^x is its very own self again: e^x

$\dfrac{d\left[\ln x\right]}{dx} = \dfrac{1}{x}$, a fact you should commit to memory

Exercises: Find the Derivative of each

i) $y = 6e^x$ ii) $f(x) = -2e^x$ iii) $y = \ln x$ iv) $y = 3\ln x$

Properties of Logarithm reminder: $\ln MN = \ln M + \ln N$ $\ln \dfrac{M}{N} = \ln M - \ln N$ $\ln M^R = R\ln M$

Use the above properties, as possible and necessary, to rewrite each function before taking the derivative:

a) $f(x) = \ln 7x$ b) $g(x) = \ln x^4$ c) $h(x) = \ln 5x^8$

We have already been putting into use the formula $A = Pe^{rt}$, which can be thought of in function form as such: $f(t) = Pe^{rt}$. This form of an Exponential function is seen and used quite often in business related mathematics. We, of course, also need its derivative much of the time. The below formula is designed for just such occasions:

$$\frac{d\left[e^{kx}\right]}{dx} = ke^{kx}$$

Use the formula to determine the derivative of each of the following functions:

i) $y = e^{2x}$

ii) $f(x) = e^{-3x}$

iii) $f(t) = e^{0.05t}$

iv) $f(t) = 500e^{-0.03t}$

EXERCISES: Use the derivative rules and Ln properties to find each function's Derivative

a) $f(x) = 4e^{x}$

b) $h(x) = \ln 3x$

c) $f(t) = e^{0.08t}$

d) $f(x) = 4e^{-2x}$

e) $g(x) = \ln x^{7} + \ln 5x$

SECTION 3.3: PRODUCT AND QUOTIENT RULES

Now that we have multiple styles of functions and their respective derivatives to work with, we need a tool to help us find the derivative when perhaps different styles of these functions are in either a product or quotient with one another. It is not as simple as when we saw them added/subtracted together.

As an opening look at a Product of two functions, suppose we have $h(x)=(x^2+3)(4x+5)$, which of course can be multiplied out by the FOIL Method. We will do so, but only after obtaining the derivative of *h(x)* by use of Product Rule first. There are multiple versions of Product Rule, among which is this:

$$\frac{d\left[f(x)\cdot g(x)\right]}{dx}=f'(x)\cdot g(x)+f(x)\cdot g'(x) \quad \text{or more briefly:} \quad \frac{d\left[f\cdot g\right]}{dx}=f'\cdot g+f\cdot g'$$

Now, when we look at $h(x)=(x^2+3)(4x+5)$, we logically select $f=x^2+3$ *and* $g=4x+5$

Next, because our formula says we need them, determine each: $f'=$ *and* $g'=$

This gives us the four items required in Product Rule. Place them in their respective spots underneath the expanded formula below, and then simplify where possible

$$f' \quad \cdot \quad g \quad + \quad f \quad \cdot \quad g'$$

$h'(x)=$

Remember, we could have used FOIL on the original *h(x)*. Do so now, what do you get?

$h(x)=$ $h'(x)=$

Take the derivative of this version of *h(x)*. Did you get a match for what Product Rule said?

Suppose now that FOIL is not possible, maybe if $h(x)=x^4e^x$. Now we MUST USE Product Rule. Do so.

$f=$ $h'(x)=$

$f'=$

$g=$

$g'=$

EXERCISES: Use Product Rule to find each derivative, simplifying afterwards when possible

i) $h(x) = x^5 \ln x$

ii) $y = 3x^2 e^{4x}$

iii) $K(x) = e^{2x} \ln x$

iv) $y = (4x^2 + 5)\sqrt{x}$

Now suppose we have *f/g* instead of them being multiplied. We have a rule for this, too, called Quotient Rule. Again, it has multiple forms, but particularly needs us to build the Numerator of the formula correctly. Use the following:

$$\frac{d\left[\dfrac{f(x)}{g(x)}\right]}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2} \quad \text{or more briefly:} \quad \frac{d\left[f/g\right]}{dx} = \frac{f'g - fg'}{\left[g\right]^2}$$

Similar to setting up the pieces for Product Rule, set up the pieces (f, f', g, g') and use them to determine the derivative of: $h(x) = \dfrac{5x+7}{x^2+3}$. SIMPLIFY THE NUMERATOR

$f =$

$h'(x) =$

$f' =$

$g =$

$g' =$

EXERCISES: Use Quotient Rule to find each Derivative

i) $h(x)=\dfrac{x^2+1}{x^2+4}$

ii) $y=\dfrac{5x-2}{x+6}$

iii) $h(x)=\dfrac{e^x}{x^2+3}$

iv) $h(x)=\dfrac{x^4}{e^{2x}}$

Mix of both: Find the derivative of $h(x)=\dfrac{x^2+1}{x^2e^x}$