SECTION 4.3: L'HOPITAL'S RULE

Sometimes when attempting to determine a Limit by the algebraic method of "plugging in the number x is approaching", we run into situations where we seem not to have an answer, but where we can dig a bit deeper and still find one. These situations are called Indeterminate, and while there are several such cases explored in the Calculus, we will only address two cases, namely 0/0 and ∞/∞.

The very most important aspect of using L'Hopital's Rule is to be certain it actually applies. Thus, the student should ALWAYS demonstrate what the Numerator and Denominator are approaching in their own separate ways. If, in fact, an Indeterminate case is NOT present, L'Hopital's cannot be used.

EXAMPLE: Find the Limit: $\lim\limits_{x\to 2}\dfrac{2x^2-8}{x^2+2x-8}$

i) Plug in the 2 and see whether you find a Real Number answer. If you do, great, that is your answer. However, here we do find one of our Indeterminate cases, the 0/0 case. So, now we know to apply L'Hopital's Rule.

ii) Take the Derivative of the Numerator and Denominator, NOT QUOTIENT RULE, and construct the new Limit we need to determine. Using proper form means writing the Limit notation, not just the new fraction with the derivatives. Then, again plug in the 2. Do you get a Real Number answer?

EXAMPLE: Find the Limit: $\lim\limits_{x\to 4}\dfrac{x^3-6x^2+32}{x^3-5x^2-8x+48}$

i) Plug in 4, show we have an Indeterminate case
ii) Apply L'Hopital's, remembering to use Limit notation
iii) Again, plug in the 4. What do you get?
iv) Can L'Hopital's be used again? (YES!!)

EXERCISES: USE L'Hopital's Rule to determine each limit. Show why it must be used before each usage.

a) $\lim_{x \to 5} \frac{x^2 - 15x + 50}{x^2 - 3x - 10}$

b) $\lim_{x \to 3} \frac{x^3 - 2x^2 - 15x + 36}{x^3 - 4x^2 - 3x + 18}$

c) $\lim_{x \to 0} \frac{e^{2x} - 1}{x^2 - x}$

d) $\lim_{x \to 0} \frac{e^{3x} - 1}{x^3 + x}$

Now let's look at the other Indeterminate case, the ∞ / ∞ case. We cannot truly "plug in ∞", but we do need to reason out whether the Numerator and Denominator do, BOTH, in fact trend towards $\pm\infty$. For the typical Polynomials we encounter in this course, the highest degree term of the polynomial determines what happens.

EXAMPLE: Find the Limit: $\lim_{x \to \infty} \frac{2x^2 - 8}{5 - 3x - 6x^2}$ What do the Numerator and Denominator do?

Of course, we get ∞ / ∞. So, apply L'Hopital's and then try again.

Do you get ∞ / ∞ a second time? No problem, apply L'Hopital's a second time. We can apply L'Hopital's as many times as needed, just keep remembering to check the limit each time.

EXERCISES: Find each Limit, applying L'Hopital's as needed

i) $\lim_{x \to \infty} \frac{12x^2 + 5x - 2}{3x^2 - 8x + 9}$

ii) $\lim_{x \to \infty} \frac{3x^2 - 8}{6x^3 + 10x - 4}$

iii) $\lim_{x \to \infty} \frac{2x^2 + 5x + 6}{3x - 6}$

SECTION 4.4: VERTICAL AND HORIZONTAL ASYMPTOTES AND GRAPHING FUNCTIONS INCLUDING THEM

Determine ALL Asymptotes of each function below:

a) $h(x) = \dfrac{2x+3}{x-4}$

b) $h(x) = \dfrac{6x+8}{x^2-4}$

c) $y = \dfrac{x+4}{x^2-16}$

d) $h(x) = \dfrac{2x}{x^2+1}$

e) $h(x) = \dfrac{x^2+8}{x-3}$

f) $y = x^2 e^x$

EXERCISE: Given the function below, perform a full Calculus-based analysis

a) Determine ALL Asymptotes

b) Take the First Derivative, and use it to determine intervals of Increase/Decrease, remembering to use any Vertical Asymptotes as “Partition Values” alongside any CP’s from $f'(x)$.

c) Identify the CPs as either a Relative Minimum, Relative Maximum, or Neither. Determine the Coordinates of each regardless of the outcome.

d) Take the Second Derivative, and use it to determine Intervals of Concavity, specifying Up or Down appropriately. Again, remember to use Vertical Asymptotes as Partition Values. Identify any Points of Inflection, in Coordinate form.

e) Sketch a graph of the function using all of the information gathered here.

$$h(x) = \frac{x}{x^2 - 4}$$

EXERCISE: Given the function below, perform a full Calculus-based analysis

a) Determine ALL Asymptotes

b) Take the First Derivative, and use it to determine intervals of Increase/Decrease, remembering to use any Vertical Asymptotes as “Partition Values” alongside any CP’s from $f'(x)$.

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e) Sketch a graph of the function using all of the information gathered here.

$$h(x) = \frac{4x}{x^2 + 1}$$

SECTION 4.5: ABSOLUTE MINIMUM AND ABSOLUTE MAXIMUM

Find the Absolute Minimum and Absolute Maximum of each function on the given closed interval

a) $f(x) = x^3 - 12x + 7 \quad on \quad [-5, 3]$

i) Find the derivative, $f'(x)$, set it = 0 and locate all Critical Points(CPs)
ii) Which CP's are "relevant", meaning within the given Interval [a , b] ?
iii) Enter the values of the Endpoints(a and b) plus the relevant CPs into the ORIGINAL f(x)
iv) Select the smallest output as the Absolute Minimum and the largest as the Absolute Maximum

b) $f(x) = x^4 + 4x^3 - 20x^2 \quad on \quad [-3, 3]$

i) Find the derivative, $f'(x)$, set it = 0 and locate all Critical Points(CPs)
ii) Which CP's are "relevant", meaning within the given Interval [a , b] ?
iii) Enter the values of the Endpoints(a and b) plus the relevant CPs into the ORIGINAL f(x)
iv) Select the smallest output as the Absolute Minimum and the largest as the Absolute Maximum

Find the Absolute Minimum and Absolute Maximum of each function, if they exist

a) $h(x) = \dfrac{x^2}{x-4} \quad on \quad (4, \infty)$

i) Find the derivative, $h'(x)$, set it = 0 and locate all Critical Points(CPs)

ii) Build a Sign Chart for the relevant intervals before and after the CP. What do the Dec/Inc Intervals suggest in terms of an Abs Min or Abs Max?

b) $h(x) = \dfrac{4x}{x^2+1} \quad on \quad (-\infty, \infty)$

i) Find the derivative, $h'(x)$, set it = 0 and locate all Critical Points(CPs)

ii) Build a Sign Chart for the relevant intervals before and after the CP. What do the Dec/Inc Intervals suggest in terms of an Abs Min or Abs Max?

Find the Absolute Minimum and Absolute Maximum of each function, if they exist, Cont'd

c) $h(x) = x^2 e^{-x} \quad on \quad (-\infty, \infty)$

i) Find the derivative, $h'(x)$, set it = 0 and locate all Critical Points(CPs)
ii) Build a Sign Chart for the relevant intervals before and after the CP. What do the Dec/Inc Intervals suggest in terms of an Abs Min or Abs Max?

d) $f(x) = 2x^4 - 8x^3 + 10 \quad on \quad (-2, 5)$

i) Find the derivative, $h'(x)$, set it = 0 and locate all Critical Points(CPs)
ii) Build a Sign Chart for the relevant intervals before and after the CP. What do the Dec/Inc Intervals suggest in terms of an Abs Min or Abs Max?

SECTION 4.6: APPLIED OPTIMIZATION

A farmer has a barn and needs to build a rectangular pen alongside it. He has 400 feet of fence for this pen, but needs to cut the pen into 3 separate areas, as shown in the diagram below right.

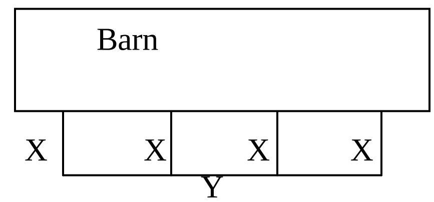
a) Build a relation between X and Y as labeled on the diagram at right

b) Solve the relation for Y, and substitute into how the Area is calculated

c) You should now have a function, A(x) representing the Area in one variable

d) Take the derivative and locate the CP. Use $A''(x)$ to show the CP is at a Maximum

e) Find the Dimensions of the pen and its total Area



EXERCISE: The Notell Motel rents out 100 rooms each night when the room rate is $60/night. Data suggests that for each $3 increase in price, they will rent out 2 less rooms. Construct a function to represent total revenues and use it to determine: (i) Maximum Revenue, (ii) the new room rate, and (iii) the number of rooms rented per night

EXERCISE: TicketMonster sells tickets to community carnivals. They notice that when the price is $12 they sell 500 tickets, but for each $1 decrease in price, they sell an extra 125 tickets. Construct a function to represent total revenues and use it to determine: (i) Maximum Revenue, (ii) the new ticket price, and (iii) the number of tickets sold.

EXERCISE: GizmoGuys sells 1200 gizmos each year, at a constant rate over the course of the year. The cost of storing one gizmo for a year is $10, while ordering costs are $15 for each order plus $25 for each gizmo.

a) Using X = Lot Size, build a complete total Cost function representing Storage and Ordering costs.
b) Use your Cost function to find a critical point, where costs are minimized
c) Determine: (i) Lot Size, (ii) # of Orders each year, and (iii) the total minimum costs