SECTION 3.4: CHAIN RULE

The next important technique for finding the Derivative of a function is Chain Rule. Similar to the Product and Quotient Rules, you must choose what "part" of the original function to declare as *f(x)* and which is *g(x)*. This is not always quite as obvious, and actually can sometimes be done in multiple ways.

First, then, some practice at decomposing some composite functions. Start with $h(x)=\left(3x+5\right)^8$, which we would like to think of in the form $h(x)=f(g(x))$ and so need to choose *f* and *g*. The first difference from Product/Quotient rules is that YOU SHOULD CHOOSE *g(x)* FIRST!! We are describing it as *f(g(x))*, where *g(x)* is clearly shown as "inside" *f*. What looks to be "inside" for this *h(x)*? Let's look at two possibilities.

i) What if we choose $g(x)=3x$? Then we ask "what is being done to *g*?". Look at *h(x)*, and if you say "5 is being added to *g(x)* and then it is raised to the 8^{th} power", we design *f(x)* to do exactly that. So, $f(x)=\left(x+5\right)^8$. The problem here is that *f(x)* is itself still a "composite function", and we really have not improved the situation.

ii) What if we choose $g(x)=3x+5$? Now, when we ask "what is being done to *g*?", a much simpler description of "it is being raised to the 8^{th} power" is our answer. So, choose $f(x)=x^8$, and this time *f(x)* is NOT a composite, but rather one of our basic derivative functions, a Power function in particular.

Your goal in choosing *g(x)* first is to make sure *f(x)* is a "simple" derivative function, and in this Business Calculus course, that means one of three cases: a Power function, $f(x)=x^n$, an Exponential function, $f(x)=e^x$, or a Logarithm, $f(x)=\ln x$. You have taken the derivative of each of these many times by now, but should do some extra practice on each just in case.

DECOMPOSITION PRACTICE: take each function below and decide first *g(x)* and then the appropriate *f(x)* from one of x^n, e^x, $\ln x$

a) $h(x)=\left(x^3+2\right)^6$

$f(x)=$

$g(x)=$

b) $h(x)=e^{4x-1}$

$f(x)=$

$g(x)=$

c) $h(x)=\sqrt{3x^2+7x}$

$f(x)=$

$g(x)=$

d) $h(x)=\ln\left(6x^2+5x\right)$

$f(x)=$

$g(x)=$

e) $h(x)=\dfrac{8}{\sqrt[3]{x^6+4}}$

$f(x)=$

$g(x)=$

f) $h(x)=e^{-3x^2}$

$f(x)=$

$g(x)=$

Now that we have practiced breaking composite functions down in a useful manner, we take a look at the general version of Chain Rule: *If* $h(x) = f(g(x))$, *then* $h'(x) = f'(g(x)) \cdot g'(x)$. Notice that similar to Product and Quotient rules, once we decided our choices of *f(x)* and *g(x)*, we need both of their derivatives, f' *and* g'. We go back to the original example, $h(x) = (3x+5)^8$, where we decided our best choices of f and g are listed below. Find both of their derivatives.

$h(x) = (3x+5)^8$

$f = x^8$ $\qquad\qquad$ $f' =$

$g = 3x+5$ $\qquad\qquad$ $g' =$

Our formula tells us to start with placing g "inside" f', and then multiply by g'. Sometimes a bit of simplifying can be done, as is the case here. Once simplified, did you get this? $h'(x) = 24(3x+5)^7$

EXERCISES: use the above process to determine the derivatives of each function

i) $h(x) = (x^2-6)^5$

$f(x) =$

$g(x) =$

ii) $h(x) = e^{5x^2+2}$

$f(x) =$

$g(x) =$

iii) $h(x) = \ln(8+3x^5)$

$f(x) =$

$g(x) =$

iv) $h(x) = \sqrt[3]{6x^2+7}$

$f(x) =$

$g(x) =$

The demonstration and exercises so far have been all about use of one single formula and concept of Chain Rule, which is meant to show that EVERY COMPOSITE FUNCTION'S DERIVATIVE CAN BE DONE IN THIS GENERAL WAY.

However, mathematics is full of many different styles of functions, and we have formulated "short-cut" formulas for Chain Rule for most of the function styles. We therefore present here a short-cut for each of the three basic function styles seen in Business Calculus.

1) For Power functions, $h(x) = \left(x^3 + 2\right)^6$, we can use General Power Rule: $\dfrac{d\left[f(x)\right]^n}{dx} = n \cdot \left[f(x)\right]^{n-1} \cdot f'(x)$
Bring the exponent out front, subtract 1 from the "old" exponent on the "inside function" and then multiply by the derivative of that inside function. Do so for *h(x)* above. Then simplify.

Did you get: $h'(x) = 18x^2\left(x^3+2\right)^5$? Try these, rewriting the function first, if necessary.

a) $h(x) = \left(2x^4 + 7\right)^5$ b) $y = \sqrt{4x^3 + 3x}$ c) $h(x) = \dfrac{3}{\left(5x^2 - 3x\right)^4}$

2) For Exponential functions, $h(x) = e^{x^2+6}$, we use $\dfrac{d\left[e^{f(x)}\right]}{dx} = e^{f(x)} \cdot f'(x)$ which tells us to quite simply first rewrite the ***original exponential function***, then multiplied by the derivative of the exponent. Do so.

$h'(x) =$

Try these: a) $h(x) = e^{9x-2}$ b) $y = e^{x^4+5x}$ c) $h(x) = e^{-2x^2}$

3) For Natural Logarithm functions, $h(x) = \ln\left(x^3 + 6x\right)$, we use: $\dfrac{d\left[\ln f(x)\right]}{dx} = \dfrac{f'(x)}{f(x)}$ which tells us to place the derivative of what is inside the ln above that "inside function": $h'(x) = \dfrac{3x^2+6}{x^3+6x}$

Try these: a) $h(x) = \ln\left(5x^3\right)$ b) $y = \ln\left(\ln x\right)$ c) $h(x) = \ln\left(x^2 + 5x\right)$

SECTION 3.5: IMPLICIT DIFFERENTIATION

We introduce one final technique for taking derivatives, which borrows help from Chain Rule and is necessary for certain situations that arise when we are unable to isolate y in a relation between x and y. Such equations are called *Implicit Equations*, so the technique is called *Implicit Differentiation*. We start with an example that does not truly need this technique, but this means we will be able to verify the correctness of the derivative we create by other means.

$$y^3 = x^2 + 5x + 4$$

Handle every variable term as if it is a composite function and needs Chain Rule, and good practice when learning this method is to write the "derivative of the inside function" in $\frac{d[\ ]}{dx}$ form. As examples, if you have the term y^2, its derivative should be written as $2y\frac{dy}{dx}$, and if you have x^2, its derivative should be written as $2x\frac{dx}{dx}$. Your understanding of fractions says that $\frac{dx}{dx}$ reduces away, but it is extremely important that in the "y terms" that a $\frac{dy}{dx}$ is present after taking the derivative of that term. The $\frac{dy}{dx}$ is your actual derivative of the overall equation once you isolate it. Back to our example, take the derivative of each term:

Do you have: $3y^2\frac{dy}{dx} = 2x\frac{dx}{dx} + 5\frac{dx}{dx} + 0$? Of course, the 0 can be ignored and the $\frac{dx}{dx}$ fractions can be reduced to 1's. You now have this:

$3y^2\frac{dy}{dx} = 2x + 5$, where we now isolate $\frac{dy}{dx}$ by dividing both sides by $3y^2$. Thus $\frac{dy}{dx} = \frac{2x+5}{3y^2}$

Going back to the original, $y^3 = x^2 + 5x + 4$, we can take Cube Roots and get: $y = \sqrt[3]{x^2 + 5x + 4}$
Use Chain Rule to obtain the derivative of this version of the function, and then use Algebra to see that both derivatives are indeed the same, bust with different forms.

Now let's add in an element of difficulty. Take the derivative of each term:

$$y^2 + 7x = x^4 + 5y + 8$$

You should have more than one instance of $\frac{dy}{dx}$, which we need to have completely isolated. A little bit of Algebra comes in handy again. Move all terms with $\frac{dy}{dx}$ to one side of the equation, and those without it to the other. This allows us to "factor out" the $\frac{dy}{dx}$ on its side of the equation, and then divide away whatever is inside the ( )'s. Do so:

Did you get one of these: $\frac{dy}{dx} = \frac{4x^3 - 7}{2y - 5}$ or $\frac{dy}{dx} = \frac{7 - 4x^3}{5 - 2y}$? Notice these are equivalent to one another if you simply multiply by -1 to both the top and bottom of the fraction. Try these:

a) $6y + x^3 - 3 = 2x^2 + y^5$

b) $e^{2y} + x^2 = e^x + 4y$

Now let us add another level of difficulty, mixing in the need for Product Rule. Look at the term with both x and y in this equation, which of course has multiplication built into it, a ***PRODUCT***. We therefore treat each portion as f and g and set up Product Rule for that term. To help you out, places for f , f' , g , g' are provided.

$$x^3 y^2 + 7 = 5x^2 + 4y$$

$f =$

$f' =$

$g =$

$g' =$

Once the tricky part of using Product Rule is done, we notice we have something rather similar to the examples on the previous page, so continue to isolate dy/dx in the way you did for those examples.

EXERCISES: Find dy/dx for each of these Implicit Equations:

i) $x^3 - y^4 + 7 = x^2 + 6y$

ii) $8x + 3 = 5x^2 y + y^2 + 2$

SECTION 3.6: RELATED RATES

Related Rates borrows the derivative techniques of Chain Rule and Implicit Differentiation to assist us in seeing how the rate of change of one variable affects the rate of change of another. For example, if the price of an item is increasing, we expect that demand for this item probably will decrease. But is the rate of increase in price small or large? One would expect different decrease levels in demand for small versus large price increases, right? This use of derivatives gives us some of that information.

Start with a relatively simple example with a geometric concept where the visuals might help you in understanding why the two ***rates of change*** are ***related*** to one another.

Suppose the fire department has a 30-ft long ladder leaning up against a wall, but because of some ice on the ground, the bottom of the ladder is sliding away from the wall at a rate of 2-ft per second. That we describe it in what can be shown with a fraction, $ft/\sec$, makes this a Rate of Change. Specifically, in math terms, ***the rate at which the distance*** from the wall the bottom of the ladder is growing ***with respect to time***, in seconds. If we give that distance from the wall the variable a, then this rate is da/dt. It is important to note the denominator is ***dt***, because this tells us our derivatives must be taken "*with respect to t*". We used *a* for the distance on the ground, let us use *b* for the height up the wall. Think about pulling a ladder's bottom away from a wall; the height where the top of the ladder touches the wall will start coming down, or decrease. How are these two distances related to one another? Picture the ladder, it forms a Right Triangle, which brings to mind the Pythagorean Theorem: $a^2+b^2=c^2$. We have already declared *a* to be the ground, *b* as the wall height, while *c* is, of course, the ladder itself. But because the ladder is a constant length of 30-ft, we place 30 in for *c* in our formula. We now have this: $a^2+b^2=900$, which formulates the relationship connecting the distance from the wall to the height up the wall.

Question: If the bottom of the ladder is sliding away from the wall at 2-ft/sec, how fast is the top of the ladder moving when the bottom of the ladder is 18 feet from the wall?

A) What do we know? $a=18$, $da/dt=2$ and we have the relation: $a^2+b^2=900$.

B) We want db/dt (the rate of change of the height up the wall), and so we take the derivative of our relation, but "*with respect to time, t*", and that means we must do it using Implicit Differentiation techniques. Do so, what do you get?

C) Isolate db/dt, what do you get?

D) We know $a=24$, plug it into $a^2+b^2=900$ to get *b*, and then plug in $a,\ b,\ and\ da/dt$ to determine our desired db/dt. Do you get $db/dt=-3/2$?

EXAMPLE: One snowy winter, a group of kids decided to build the biggest snowman ever. They started rolling up a snowball in a field, pushing it around and around to form a perfect sphere, eventually reaching a size where the sphere had a radius of 100cm(for those unsure of the metric system, this would be about 6 feet, 7 inches in height). They rolled up a second snowball, but were unable to figure out how to mount it atop the first snowball, and so abandoned the snowman project. Springtime soon followed and the giant 100cm radius snowball began to melt, the radius shrinking at a rate of 2cm/day. When the radius is down to 75cm, how much volume of water is melting off the snowball per day?

A) What information do we know? Assign variables as necessary. Also decide what *rate* are we looking to determine, in notational form(for example, db/dt as on the previous page)?

B) What can we use as a *relation* between variables? A geometry formula will help again.

C) Take the derivative of your formula "***with respect to time, t***".

D) Do you have everything you need to plug in and solve for our desired rate? If so, do so, and if not, obtain any missing values by use of the relation formula itself(as we did on the previous page).

Did you get : $dV / dt = 141372\, cm^3 / day$, rounded to the nearest whole number. For those curious, this translates to about 141.4 liters per day or roughly 37.4 gallons of water per day.

EXERCISE: Suppose that the Quantity, x in 1000's, and Price, p in \$, are related by the Price-Demand equation given here:

$$2p^2 + 6xp + 3x^2 = 650$$

i) If the Price is changing at a rate of \$3/month, at what rate is the Quantity demanded changing when the Price is \$10?

a) We know $p = 10$ *and* $dp/dt = 3$ but not x. Substitute $p = 10$ into the equation and then use the Quadratic Formula to verify you get $x = -10 + 5\sqrt{10} \approx 5.8$

b) Now take the derivative Implicitly, with respect to time t. Solve it for dx/dt and then plug in the known values to determine dx/dt.

ii) What if instead we wish to know the rate of change of price when the change in demand is 4000(remember x is in 1000's) and the quantity is currently 10000? (Hint: you might need Quadratic Formula again, but it should be factorable this time)

SECTION 3.7: ELASTICITY OF DEMAND

The construction and underlying logic that leads to the formula below was given in class, so we do not present it again here. We present some examples using the formula to build the Elasticity function, and then use it.

$$E(p) = \frac{-p \cdot f'(p)}{f(p)}$$

EXAMPLE: If the demand is given by $x = f(p) = 500 - 8p$, build the Elasticity function and then determine the Elasticity when (i) p = 20 and (ii) p = 40

a) To build *E(p)*, first find $f'(p)$, place it into the formula and simplify the new function.

b) Plug in each price: (i) 20 and (ii) 40

EXAMPLE: If the demand is given by $x = f(p) = 6000 - 5p^2$, build the Elasticity function and then determine the Elasticity when (i) p = 25, (ii) p = 15

Let us continue looking at the last example, $x = f(p) = 500 - 8p$, and the idea of Unit Elasticity, or when $E(p) = 1$. Unit Elasticity suggests that a small change in price will have a similarly small effect on the demand, and that total revenues cannot be improved. In other words, that revenue is at a maximum.

You should have found the Elasticity function to be $E(p) = \dfrac{10p^2}{6000 - p^2}$. Set the function equal to +1 and solve for the value of p which satisfies this condition.

EXERCISE: If the demand is given by $x = f(p) = 800 - 4p$, build the Elasticity function and then determine the Elasticity when (i) p = 250, (ii) p = 75. Also determine at what value p we find Unit Elasticity by the method described on the top half of this page.