MATH 180 Final Exam

December 14, 2017

Directions. Fill in each of the lines below. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR INSTRUCTOR. This exam contains 12 pages (including this cover page) and 15 problems. After starting the exam, check to see if any pages are missing. Enter all requested information on this page. You are expected to abide by the University's rules concerning Academic Honesty.

TA Name:________________________

Answers

The following rules apply:

- You may *not* use your books, notes, calculators, or any electronic device including cell phones. Only pencils/pens allowed.
- You must show all of your work. An answer, right or wrong, without the proper justification will receive little to no credit.
- You *must* complete your work in the space provided. We will be scanning your answers into our grading system, so any work you do that is out of place, too close to the page border, or on the wrong page will *not* be graded!

Circle your instructor.

- Martina Bode
- Jenny Ross
- Drew Shulman
- Matthew Woolf
- Sherwood Hachtman

DO NOT WRITE ABOVE THIS LINE!!

1. (24 points) Find the derivatives of the following functions. Use logarithmic differentiation if necessary. You do not need to simplify your answers.

(a) (6 points) $f(x) = x^5 + \sqrt[3]{x} + \tan(4x)$

$$f'(x) = 5x^4 + \frac{1}{3}x^{-2/3} + 4\sec^2(4x)$$

(b) (8 points) $g(x) = \dfrac{\arctan(2x)}{e^{3x}+4}$

$$g'(x) = \frac{\frac{2}{1+(2x)^2}(e^{3x}+4) - \arctan(2x) \cdot 3e^{3x}}{(e^{3x}+4)^2}$$

(c) (10 points) $k(x) = (x^2+4)^x$

$$\ln k(x) = x \ln(x^2+4)$$

$$\frac{1}{k(x)} \cdot k'(x) = 1 \cdot \ln(x^2+4) + x \cdot \frac{2x}{x^2+4}$$

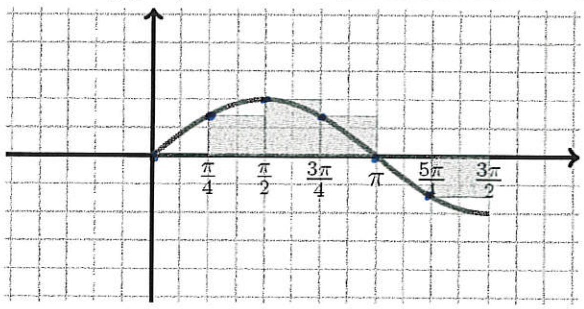
$$k'(x) = \left(\ln(x^2+4) + \frac{2x^2}{x^2+4}\right)(x^2+4)^x$$

2. (8 points) Fill in the blanks for the following statements.

(a) (4 points) If $\int_2^6 f(x)dx = 8$, then the average value of f on the interval $[2, 6]$ is 2.

ave $= \frac{1}{b-a} \int_a^b f(x)dx = \frac{1}{6-2} \cdot 8 = \frac{1}{4} \cdot 8 = 2$

(b) (4 points) The following picture illustrates a Left-endpoint Riemann sum on the interval $[0, 3\pi/2]$ with $n =$ 6 subintervals, and $\Delta x =$ $\pi/4$.



3. (12 points) Let $f(x) = 3x^2 - 4$. **Use the limit definition** of the derivative to compute $f'(x)$. No credit will be given to the correct answer without the limit computation!

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(3(x+h)^2 - 4) - (3x^2 - 4)}{h}$$

$$= \lim_{h\to 0} \frac{3(\cancel{x^2} + 2xh + h^2) \cancel{-4} - \cancel{3x^2} \cancel{+4}}{h}$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2}{h}$$

$$= \lim_{h\to 0} 6x + 3h = 6x$$

4. (10 points) Let

$$f(x) = \begin{cases} 2\cos x & if \quad x \le 0 \\ x^2 + 2 & if \quad 0 < x < 1 \\ 7x + \pi & if \quad x \ge 1 \end{cases}$$

(a) (3 points) Evaluate the following limits:

$\lim_{x\to 0^-} f(x) = 2$ $= \lim_{x\to 0} 2\cos x = 2$

$\lim_{x\to 0^+} f(x) = 2$ $= \lim_{x\to 0} x^2+2 = 2$

$\lim_{x\to 0} f(x) = 2$

(b) (2 points) Is f continuous at $x = 0$?

Yes, because $\lim_{x\to 0} f(x) = 2 = f(0)$

(c) (3 points) Evaluate the following limits:

$\lim_{x\to 1^-} f(x) = 3$ $\quad$ $\lim_{x\to 1^+} f(x) = 7+\pi$ $\quad$ $\lim_{x\to 1} f(x)$ does not exist

(d) (2 points) Is f continuous at $x = 1$?

No, because $\lim_{x\to 1} f(x)$ does not exist.

5. (12 points) Use implicit differentiation to find an equation of the tangent line to $x^2 + 4y^2 - 2xy = 25$ at the point $(-5, 0)$.

$$2x + 8y \cdot \frac{dy}{dx} - \left(2y + 2x\frac{dy}{dx}\right) = 0$$

@ $x = -5$ & $y = 0$

$$-10 + 0 - \left(0 - 10\frac{dy}{dx}\right) = 0$$

$+10 \qquad +10$

$$-10\frac{dy}{dx} = +10$$

$$\frac{dy}{dx} = -1$$

$\Rightarrow y - 0 = -1(x+5)$

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6. (20 points) Find the following limits. Clearly indicate if a limit goes to $+\infty$ or $-\infty$. Show all your work and use L'Hôpital's rule where appropriate!

   (a) (5 points) $\lim_{x\to 2} \frac{\sqrt{x+2}-2}{x^2-4}$ of type $\frac{0}{0}$

$$= \lim_{x\to 2} \frac{(\sqrt{x+2}-2)(\sqrt{x+2}+2)}{(x+2)(x-2)\cdot(\sqrt{x+2}+2)}$$

$$= \lim_{x\to 2} \frac{\cancel{x-2}}{(x+2)(\cancel{x-2})(\sqrt{x+2}+2)} = \frac{1}{4\cdot 4} = \boxed{\frac{1}{16}}$$

   (b) (4 points) $\lim_{x\to 3^+} \frac{2}{x-3}$ of type $\frac{2}{0^+}$

$$= \boxed{+\infty}$$

   (c) (5 points) $\lim_{x\to\infty} \frac{3x+e^{2x}}{e^{4x}}$ of type $\frac{\infty}{\infty}$

$$\underset{L'H}{=} \lim_{x\to\infty} \frac{3+2e^{2x}}{4e^{4x}} \quad \text{of type } \frac{\infty}{\infty}$$

$$\underset{L'H}{=} \lim_{x\to\infty} \frac{4e^{2x}}{\underset{4}{\cancel{16}}e^{4x}} = \lim_{x\to\infty} \frac{1}{4}\cdot\frac{1}{e^{2x}} = \boxed{0}$$

   (d) (6 points) $\lim_{x\to\infty} (3x)^{1/x}$ of type ∞^0

$$L = \lim_{x\to\infty} (3x)^{1/x}$$

$$\ln L = \lim_{x\to\infty} \frac{1}{x}\ln(3x)$$

$$= \lim_{x\to\infty} \frac{\ln(3x)}{x} \quad \text{"}\frac{\infty}{\infty}\text{"}$$

$$\underset{L'H}{=} \lim_{x\to\infty} \frac{\frac{3}{3x}}{1} = 0 \quad \Rightarrow \quad L = e^0 = \boxed{1}$$

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7. (10 points) Suppose a continuous function f has derivatives:

$$f'(x) = \ln\left(\frac{x^2}{2} + \frac{1}{2}\right) \text{ and } f''(x) = \frac{2x}{x^2+1}$$

When is f concave up? When is f concave down? Does f have any points of inflection, if yes for which x-values?

sign of f'' $\quad - \quad | \quad +$

$\qquad\qquad\qquad\quad 0$

$f''=0 \quad x=0$

$f'=\phi$ N/A

concave down for $x<0$

concave up for $x>0$

point of inflection for $x=0$

8. (10 points) Consider the function $f(x) = x^3 + 3x^2 - 9x - 10$. Find the absolute maximum and absolute minimum of f on the interval $[0, 3]$.

$$f'(x) = 3x^2 + 6x - 9$$
$$= 3(x^2 + 2x - 3)$$
$$= 3(x+3)(x-1)$$

(i) $f'=0 \quad \cancel{x=-3}$ not in $[0,3]$! $\quad x=+1$

(ii) $f'=\phi$ N/A

(iii) endpoints $x=0$ & $x=3$

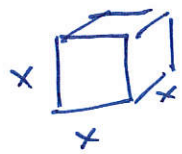
Compare values

x	$f(x)$
0	-10
1	-15 = min
3	$54-27-10=17$ = max

9. (12 points) Sally places a marshmallow in the microwave to make a S'more. The marshmallow has the shape of a **cube**.

When she turns on the microwave, the volume of the marshmallow starts to expand at the rate of 2 cm^3 per second. Assume the marshmallow stays in the shape of a cube as it grows.

Find the rate at which the length of the sides is increasing when the sides are 4 cm. Include units in your answer.



$V = x^3$

$\frac{dV}{dt} = 2$

$\frac{dx}{dt} = ?$ when $x = 4$

$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

$2 = 3 \cdot 4^2 \cdot \frac{dx}{dt}$

$\frac{dx}{dt} = \frac{\cancel{2}}{3 \cdot \cancel{16}_{8}} = \frac{1}{24}$ cm/second

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10. (12 points) Use linear approximation to estimate $\sqrt[3]{8.12}$. Is your approximation an overestimate or an underestimate? Explain!

$f(x) = \sqrt[3]{x} \qquad a = 8$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f(8) = 2$

$f'(8) = \frac{1}{3}\frac{1}{(\sqrt[3]{8})^2} = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}$

$L(x) = f(8) + f'(8)(x-8)$

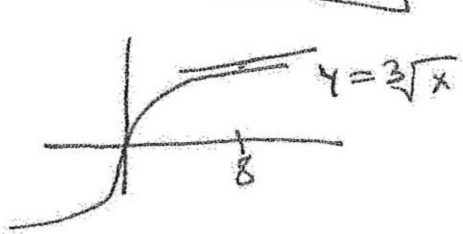
$L(x) = 2 + \frac{1}{12}(x-8)$

$\sqrt[3]{8.12} \approx 2 + \frac{1}{12}(8.12-8)$

$= 2 + \frac{1}{12}(.12)$

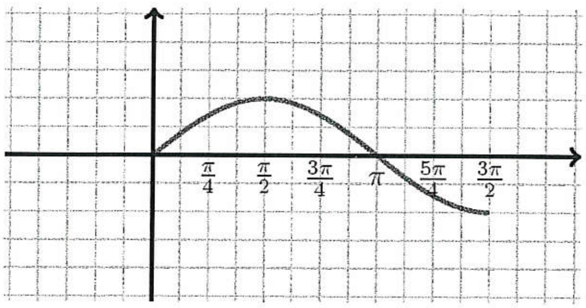
$= 2 + .01$

$= \boxed{2.01}$



OVER ESTIMATE

11. (18 points) Let $A(x) = \int_0^x \sin t \, dt$ for $0 \le x \le 3\pi/2$. The graph of $y = \sin t$ is given below.



(a) (6 points) Evaluate

(i) $A(0) = 0$

$= \int_0^0 \sin t \, dt = 0$

(ii) $A(\pi/2) = 1$

$= \int_0^{\pi/2} \sin t \, dt$

$= -\cos t \Big|_0^{\pi/2}$

$= -\cos \frac{\pi}{2} + \cos 0$

$= 0 + 1 = 1$

(iii) $A(\pi) = 2$

$= \int_0^{\pi} \sin t \, dt$

$= -\cos t \Big|_0^{\pi}$

$= -\cos \pi + \cos 0$

$= -(-1) + 1 = 2$

(b) (6 points) Find the derivatives:

(i) $A'(x) = \sin x$

FTC

(ii) $A'(\pi)$

$= \sin \pi = 0$

(iii) $A''(x)$

$= (\sin x)' = \cos x$

(c) (6 points) When is $A(x)$ increasing? decreasing? Classify the critical point(s).

increasing $0 \le x < \pi$

decreasing $\pi < x < \frac{3\pi}{2}$

@ $x = \pi$ local max (actually a global max)

(Note @ $x = 0$ global min, and nothing @ $x = \frac{3\pi}{2}$ since endpoints cannot be local max/min per definition)

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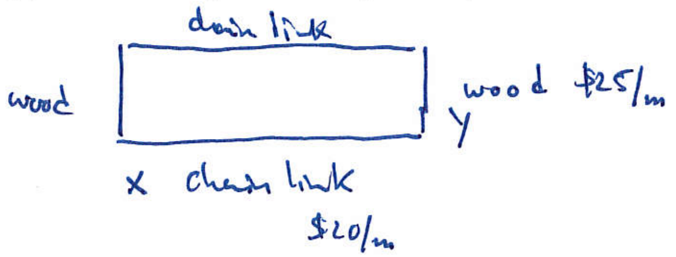
12. (14 points) Farmer Terry wants to build a fence enclosing a rectangle with area 20 m^2. Two parallel sides will be wood, which costs \$25 per meter, and the other two sides will be chain link, which costs \$20 per meter. What dimensions minimize the cost of the fence?

To do so, answer the following.

(a) Sketch a diagram of the problem, define variables to be used and carefully label your picture.



(b) Find the objective (optimization) function. Include the domain.

$$\text{Cost} = 2 \cdot 20x + 2 \cdot 25y$$

$$\text{Subject to } xy = 20 \Rightarrow y = \frac{20}{x}$$

$$f(x) = \text{cost} = 40x + 50 \cdot \frac{20}{x}$$

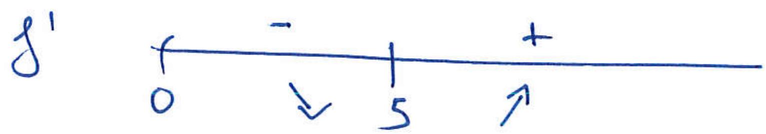
$$f(x) = 40x + \frac{1000}{x} \qquad \text{Domain} = (0, \infty)$$

(c) Find the dimensions that minimize the cost, and verify that your result gives the minimum possible cost.

$$f'(x) = 40 - \frac{1000}{x^2} = 0$$

$$40 = \frac{1000}{x^2}$$

$$x^2 = \frac{1000}{40} = 25 \Rightarrow x = \pm 5 \Rightarrow x = 5$$



min cost when $x = 5_m$ & $y = \frac{20}{5} = 4_m$

13. (16 points) Evaluate the following expressions:

(a) (6 points) $\int_1^3 (3x^2 + 5x + 2)\, dx$

$= x^3 + \frac{5}{2}x^2 + 2x \Big|_1^3$

$= \left(27 + \frac{5}{2} \cdot 9 + 6\right) - \left(1 + \frac{5}{2} + 2\right)$

$= 30 + \frac{45 - 5}{2} = 30 + 20 = 50$

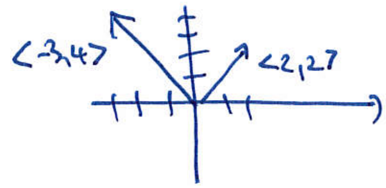
(b) (4 points) $\int \left(\frac{2}{x^2+1} + \frac{3}{x}\right) dx$

$= 2 \arctan x + 3 \ln|x| + c$

(c) (6 points) $\frac{d}{dx}\left(\int_1^{x^3} \sin(t^2+1)\, dt\right)$ use FTC + CHAIN RULE

$= \sin\left((x^3)^2 + 1\right) \cdot 3x^2$

14. (12 points) (a) (2 points) Draw the vectors $\langle -3, 4\rangle$ and $\langle 2, 2\rangle$.



(b) (6 points) Find the projection vector of the vector $\langle -3, 4\rangle$ onto the vector $\langle 2, 2\rangle$.

$$\text{proj}_{\langle 2,2\rangle} \langle -3,4\rangle = \frac{\langle -3,4\rangle \cdot \langle 2,2\rangle}{\langle 2,2\rangle \cdot \langle 2,2\rangle} \cdot \langle 2,2\rangle = \frac{-6+8}{4+4}\langle 2,2\rangle = \frac{2}{8}\langle 2,2\rangle = \frac{1}{4}\langle 2,2\rangle$$
$$= \langle \tfrac{1}{2}, \tfrac{1}{2}\rangle$$

(c) (4 points) Find the cosine of the angle between the vectors $\langle -3, 4\rangle$ and $\langle 2, 2\rangle$.

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cdot\cos\theta$$
$$\Rightarrow \cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\cdot\|\vec{b}\|} = \frac{2}{5\cdot\sqrt{8}}$$

15. (10 points) Hänsel and Gretel are pulling a sled full of candy horizontally along the ground. Hänsel and Gretel are pulling the sled with a combined force of 20 N at an angle of $\pi/6$ to the ground.

If they pull the sled 10 m, find the work done on the sled.

$$\text{Work} = \vec{F}\cdot\vec{D}$$
$$= \|\vec{F}\|\cdot\|\vec{D}\|\cdot\cos\theta$$
$$= 20\cdot 10\cdot\cos\pi/6$$
$$= 200\,\frac{\sqrt{3}}{2}$$
$$= 100\sqrt{3}\ \text{Nm or Joules}$$