MATH 180 Exam 1

February 20, 2018

Directions. Fill in each of the lines below. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR EXAM PROCTOR. This exam contains 8 pages (including this cover page) and 10 problems. After starting the exam, check to see if any pages are missing. Enter all requested information on this page. You are expected to abide by the University's rules concerning Academic Honesty.

TA Name:________________________

The following rules apply:

- You may *not* use your books, notes, calculators, or any electronic device including cell phones. Only pencils/pens allowed.
- You must show all of your work. An answer, right or wrong, without the proper justification will receive little to no credit.
- You *must* complete your work in the space provided. We will be scanning your answers into our grading system, so any work you do that is out of place, too close to the page border, or on the wrong page will *not* be graded!

Circle your instructor.

- Martina Bode
- Gary (Clark) Alexander
- John Steenbergen
- Daniel Braithwaite

1. (12 points) A function f is defined by $f(x) = \begin{cases} x^2+1 & \text{if } x \leq 3 \\ -4x+B & \text{if } x > 3 \end{cases}$

(a) (4 points) Evaluate:

(i) (2 points) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3}(x^2+1) = 9+1 = 10$

(ii) (2 points) $\lim_{x\to 3^+} f(x) = \lim_{x\to 3}(-4x+B) = -12+B$

(b) (3 points) For what value of B is $f(x)$ continuous at $x = 3$? Justify your answer.

when $10 = -12+B \Rightarrow \boxed{B=22}$

then $\lim_{x\to 3^-} f(x) = 10 = \lim_{x\to 3^+} f(x) = 10$

$\Rightarrow \lim_{x\to 3} f(x) = 10 = f(3)$

(c) (5 points) What is the equation of the line tangent to $f(x)$ at $x = 2$?

$f'(2) = 2\cdot 2 = 4 = \text{slope}$

point $(2,5)$

$\Rightarrow y-5 = 4(x-2) \Rightarrow \boxed{y = 4x-3}$

2. (6 points) Find the points on the curve $y = x^3 + x$ where the slope of the tangent line is 4.

slope $= y' = 3x^2+1 = 4$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

$\Rightarrow$ $x=1$ $y=2$

$x=-1$ $y=-2$

$\Rightarrow \boxed{(1,2) \ \& \ (-1,-2)}$

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3. (12 points) Evaluate the following limits. If the limit is infinite, state whether it is ∞ or $-\infty$. Clearly explain your reasoning, stating theorems as needed.

(a) (6 points) $\lim_{x\to 1} \frac{x^2-3x+2}{x^2-4x+3}$

$$= \lim_{x\to 1} \frac{(x-2)\cancel{(x-1)}}{(x-3)\cancel{(x-1)}}$$

$$= \frac{1-2}{1-3} = \frac{-1}{-2} = \frac{1}{2}$$

(b) (6 points) $\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}$ of type $\frac{0}{0}$

$$= \lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3}$$

$$= \lim_{x\to 9} \frac{\cancel{(x-9)}(\sqrt{x}+3)}{\cancel{x-9}}$$

$$= \sqrt{9}+3$$

$$= 6$$

4. (16 points) Let $f(x) = \dfrac{x}{x^2+6x-7} = \dfrac{x}{(x+7)(x-1)}$

(a) (13 points) Evaluate the following limits.

(i) (5 points) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{\frac{1}{x}}{1+\frac{6}{x}-\frac{7}{x^2}} = \dfrac{0}{1+0+0} = 0$

(ii) (2 points) $\lim_{x\to 3} f(x) = \dfrac{3}{9+18-7} = \dfrac{3}{20}$

(iii) (3 points) $\lim_{x\to -7^-} f(x) = -\infty$ of type $\dfrac{-7}{0^-\cdot(-8)}$

(iv) (3 points) $\lim_{x\to 1^+} f(x) = +\infty$ of type $\dfrac{1}{8\cdot 0^+}$

(b) (3 points) Find the horizontal and vertical asymptotes (if any exist) for the function $f(x)$.

$y=0$ Horizontal asymptote, see (i)

$x=-7$ Vertical asymptote, see (iii)

$x=1$ Vertical asymptote, see (iv)

5. (5 points) Suppose a rocket launches vertically and its height in meters after t seconds is given by

$$h(t) = 3t^2 + 2t$$

What is the instantaneous velocity of the rocket at time $t = 3$?

$v(t) = 6t + 2$

$v(3) = 18 + 2 = 20 \text{ m/s}$

6. (10 points) State and use the limit definition to compute the derivative function $f'(x)$ for the function:

$$f(x) = \frac{1}{2x-1}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \left(\frac{1}{2(x+h)-1} - \frac{1}{2x-1} \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \left(\frac{\cancel{2x-1} - (\cancel{2x}+2h\cancel{-1})}{(2(x+h)-1)(2x-1)} \right)$$

$$= \lim \frac{1}{\cancel{h}} \cdot \frac{-2\cancel{h}}{(2(x+h)-1)(2x-1)}$$

$$= \boxed{\frac{-2}{(2x-1)^2}}$$

7. (7 points) In this problem you will use the Squeeze Theorem to evaluate the limit $\lim_{x\to 0+} x^3\cos(1/x)$.

(a) (2 points) Fill in the blanks for the Squeeze Theorem:

For functions f, g, and h.

If $g(x) \leq f(x) \leq h(x)$ for x near a point a, but not necessarily at a, then:

$$\lim_{x\to a} g(x) \leq \lim_{x\to a} f(x) \leq \lim_{x\to a} h(x).$$

(b) (5 points) Consider $f(x) = x^3\cos(1/x)$:

Since $-1 \leq \cos(1/x) \leq 1$ for all ~~$x \neq 0$~~, $x > 0$

then $-x^3 \leq x^3\cos(1/x) \leq x^3$ for all ~~$x \neq 0$~~ $x > 0$

It follows from the Squeeze Theorem:

$$0 = \lim_{x\to 0^+} (-x^3) \leq \lim_{x\to 0+} x^3\cos(1/x) \leq \lim_{x\to 0^+} x^3 = 0$$

Therefore: $\lim_{x\to 0+} x^3\cos(1/x) = 0$

8. (6 points) Let $f(x) = \sin\left(\frac{x}{2}\right) + x + 1$. We will use the Intermediate Theorem to prove that $f(x)$ is exactly equal to 2 for some value of x between 0 and π.

Fill in the blanks:

$f(x) = \sin\left(\frac{x}{2}\right) + x + 1$ is a continuous function on the interval $[0, \pi]$.

$L = 2$ is a number that is strictly between $f(0) = 1$ and $f(\pi) = \pi + 2$

Then by the Intermediate Value Theorem there exists at least one number c in the interval (0, π) satisfying $f(c) = 2$.

This proves that $f(x) = 2$ has a solution between 0 and π.

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9. (18 points) Find the derivatives of the following functions. Do not simplify your answers.

(a) (6 points) $f(x) = \sin(x)\cos(3x)$

$$f'(x) = \cos x \cdot \cos(3x) + \sin x \cdot (-3\sin(3x))$$

(b) (6 points) $g(x) = \dfrac{7x}{\tan x}$

$$g'(x) = \frac{7\tan x - 7x \cdot \sec^2 x}{\tan^2 x}$$

(c) (6 points) $h(t) = \left(\cos(\pi t)\right)^3$

$$h'(t) = 3(\cos(\pi t))^2 \cdot (-\sin(\pi t)) \cdot \pi$$

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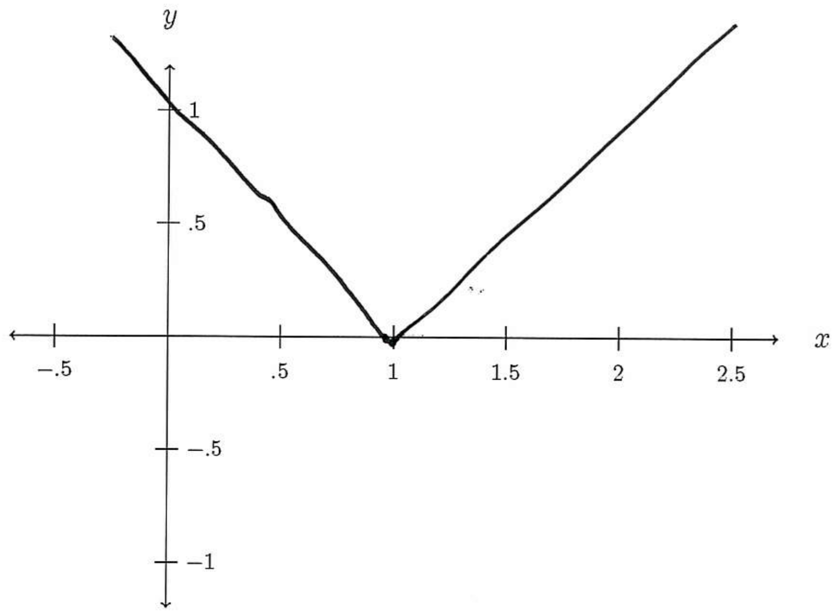
10. (8 points) A function g is defined by $g(x) = |x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ -x+1 & \text{if } x < 1 \end{cases}$

(a) (2 points) Sketch the graph of $g(x)$.



(b) (6 points) Sketch the graph of the derivative $g'(x)$.