MATH 180 Exam 2

October 30, 2018

Directions. Fill in each of the lines below. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR EXAM PROCTOR. This exam contains 8 pages (including this cover page) and 9 problems. After starting the exam, check to see if any pages are missing. Enter all requested information on this page. You are expected to abide by the University's rules concerning Academic Honesty.

TA Name:________________________

Answers

The following rules apply:

- You may *not* use your books, notes, calculators, or any electronic device including cell phones. Only pencils/pens allowed.
- You must show all of your work. An answer, right or wrong, without the proper justification will receive little to no credit.
- You *must* complete your work in the space provided. We will be scanning your answers into our grading system, so any work you do that is out of place, too close to the page border, or on the wrong page will *not* be graded!

Circle your instructor.

- Martina Bode
- Mercer (Tabes) Bridges
- Nathan Jones
- Matthew Lee
- John Steenbergen

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1. (10 points) Find an equation of the line tangent to the curve $x^2 + xy - y^3 = 7$ at $(3, 2)$.

implicit differentiation

$$2x + \underbrace{1 \cdot y + x \cdot \frac{dy}{dx}}_{\text{product rule}} - 3y^2 \cdot \frac{dy}{dx} = 0$$

@ (3,2)

plug in $x=3$ and $y=2$

$$2 \cdot 3 + 2 + 3\frac{dy}{dx} - 3 \cdot 4 \frac{dy}{dx} = 0$$

$$8 - 9\frac{dy}{dx} = 0$$

$$8 = 9\frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{8}{9}$$

$\Rightarrow$ tangent $\boxed{y - 2 = \frac{8}{9}(x-3)}$

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2. (22 points) Differentiate the following functions, use logarithmic differentiation if needed. You do not need to simplify your answers.

(a) (6 points) $f(x) = \ln(x^8 + x^4 + 5)$

$$f'(x) = \frac{1}{x^8 + x^4 + 5} \cdot (8x^7 + 4x^3)$$

(b) (6 points) $g(x) = \arctan(x^2 - 4)$

$$g'(x) = \frac{1}{(x^2-4)^2+1} \cdot 2x$$

(c) (10 points) $h(x) = (x^2 + 1)^{3x}$ log differentiation

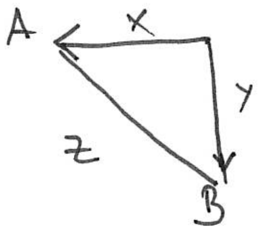
$$\ln h(x) = 3x \cdot \ln(x^2+1)$$

$$\frac{1}{h(x)} \cdot h'(x) = 3 \cdot \ln(x^2+1) + 3x \cdot \frac{2x}{x^2+1}$$

$$h'(x) = \left(3\ln(x^2+1) + \frac{6x^2}{x^2+1}\right) \cdot (x^2+1)^{3x}$$

3. (12 points) Two boats leave a port at the same time. Boat A travels west at 16 mph, and boat B travels south at 12 mph.

(a) (4 points) After half an hour, how far is each boat from the port?



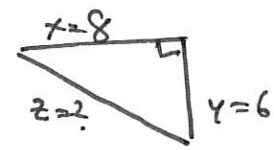
Given: $\frac{dx}{dt} = 16$ & $\frac{dy}{dt} = 12$

@ $t = \frac{1}{2}$

Boat A is 8 miles from the port, and boat B is 6 miles.

(b) (8 points) At what rate is the distance between the boats changing half an hour after they leave the port? Show all your work!

Want: $\frac{dz}{dt}$ @ $t = \frac{1}{2}$, i.e. $x = 8$ & $y = 6$



$z = 10$

$$x^2 + y^2 = z^2$$

$$\Rightarrow \quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$2\cdot(8)(16) + 2\cdot(6)(12) = 2(10)\frac{dz}{dt}$$

$$\Rightarrow \quad \frac{dz}{dt} = \frac{\cancel{2}\cdot(8)(16) + \cancel{2}(6)(12)}{\cancel{2}\cdot(10)} = \frac{200}{10} = 20 \text{ mph}$$

the distance is increasing @ 20 mph.

4. (10 points) Assume that the **derivative** of $g(x)$ is given by $g'(x) = (x+3)(x^2+1)e^x$.
On what interval(s) is g increasing? decreasing? At what value(s) of x does g have a local maximum? local minimum?

$g' = 0 \Rightarrow x = -3$
$g' = \emptyset$ does not apply

$\Big\}\Rightarrow x=-3$ is the only critical point.

sign of g': $-$ on the left of -3 ($\searrow$), $+$ on the right of -3 ($\nearrow$)

g is increasing on $(-3, \infty)$
decreasing on $(-\infty, -3)$

@ $x = -3$ g has a local minimum

no local max

5. (12 points) Find the absolute maximum and minimum values of the function $f(x) = 24x^4 - 16x^3$ on the interval $[0, 1]$. For which x-values do they occur?

$f'(x) = 98x^3 - 48x^2$

(i) $f' = 0 \Rightarrow 48(2x^3 - x^2) = 0$

$2x^2(x - \frac{1}{2}) = 0$

$x = 0$ or $x = \frac{1}{2}$

(ii) $f' = \emptyset$ N/A

(iii) endpoints $x = 0$ & $x = 1$

Compare values

x	$f(x)$
0	0
$\frac{1}{2}$	$-\frac{1}{2}$ = absolute min
1	8 = absolute max

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6. (4 points) Consider the function $f(x) = 1 - |x|$ on $[-1, 1]$, then circle the statement that is correct. Note that only one of the statements is correct.

   (a) Since $f(-1) = f(1)$, we can apply Rolle's Theorem, and thus there is a c in $(-1, 1)$ with a horizontal tangent line at $x = c$.

   (b) Since f is continuous on $[-1, 1]$, by the Mean Value Theorem there is a c in $(-1, 1)$ with:
   $$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$$

   (c) The conditions of the Mean Value Theorem are not satisfied since f is not differentiable at $x = 0$. Also the derivative is either 1 or -1, but never 0, thus the Mean Value Theorem does not apply to the interval $[-1, 1]$.

   (d) The conditions of the Mean Value Theorem are not satisfied since f is not differentiable at $x = 0$ but there is a c in $(-1, 1)$ with:
   $$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$$

7. (12 points) Let $f(x) = x^4 - 2x^3$. On what intervals is f concave up? concave down? At what value(s) of x does f have a point of inflection.

$f'(x) = 4x^3 - 6x^2$

$f''(x) = 12x^2 - 12x$

$= 12x(x-1)$

(i) $f'' = 0 \Rightarrow x = 0$ or $x = 1$

(ii) $f'' = \phi$ N/A

Sign of f'': $+$ | 0 | $-$ | 1 | $+$

$\cup$ $\cap$ $\cup$

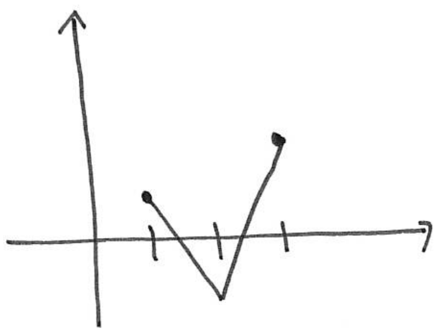
concave up on $(-\infty, 0)$ and $(1, \infty)$

concave down on $(0, 1)$

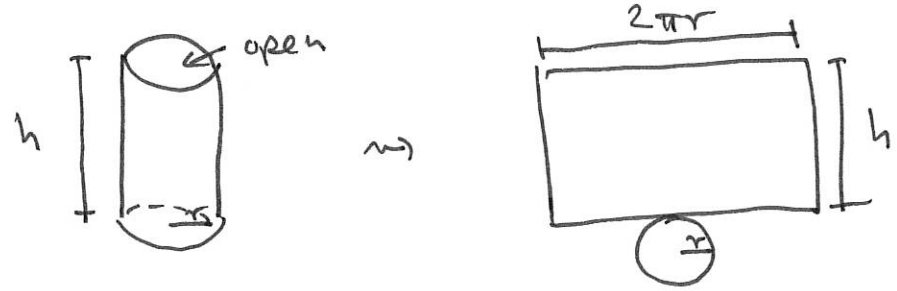
Two points of inflection: @ $x = 0$ & @ $x = 1$

8. (6 points) Sketch a graph of a function satisfying the following conditions:

   f is defined and continuous on the closed interval $[1, 3]$, while f' is only defined on $[1, 2)$ and $(2, 3]$. The absolute maximum is at $x = 3$, and the absolute minimum is at $x = 2$.



9. (12 points) A manufacturer needs to make a cylindrical can with an open top that will hold 2 liters of liquid. Determine the radius of the can that will minimize the amount of material used in its construction. Show all your work (this includes finding a domain for your optimization function), and make sure to justify your answer.



Find minimum surface area subject to volume = 2 liters.

Surface area = $2\pi r h + \pi r^2$

Volume $= \pi r^2 \cdot h = 2$

$\Rightarrow h = \dfrac{2}{\pi r^2}$

$\Rightarrow f(r) = 2\pi r \cdot \dfrac{2}{\pi r^2} + \pi r^2$

$f(r) = \dfrac{4}{r} + \pi r^2$ Domain $= (0, \infty)$

$f'(r) = -\dfrac{4}{r^2} + 2\pi r$

$f' = 0 \Rightarrow 2\pi r = \dfrac{4}{r^2} \Rightarrow r^3 = \dfrac{4}{2\pi} = \dfrac{2}{\pi} \Rightarrow r = \sqrt[3]{\dfrac{2}{\pi}}$

$f' = \emptyset$ N/A ($r=0$ is not in the domain)

Sign of f': on $(0, \sqrt[3]{2/\pi})$: $-$ ↘ ; on $(\sqrt[3]{2/\pi}, \infty)$: $+$ ↗

absolute min @ $r = \sqrt[3]{\dfrac{2}{\pi}}$

$\Rightarrow$ The amount of material is minimized for $r = \sqrt[3]{\dfrac{2}{\pi}}$