DO NOT WRITE ABOVE THIS LINE!!

Answers

MATH 181 Final Exam
May 4, 2017

Directions. Fill in each of the lines below. Circle your instructor's name and write your TA's name. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR INSTRUCTOR. Good luck!

Print Name: ______________________________

University Email: ______________________________

UIN: ______________________________

Circle your instructor's name: Boester Cappetta Steenbergen

TA's Name: ______________________________

- VERY IMPORTANT!!! CHECK THAT THE NUMBER AT THE TOP OF EACH PAGE OF YOUR EXAM IS THE SAME. IT IS THE NUMBER PRECEDED BY A POUND (#) SIGN. IF THEY ARE NOT ALL THE SAME, NOTIFY YOUR INSTRUCTOR OR TA RIGHT AWAY.
- All of your work must fit within the boxes on each page for each question. Nothing outside of the box will be graded! If you write outside of the box, there is a good chance that your exam will not be read and therefore not graded.
- A solution for one problem may not go on another page.
- Make clear to the grader what your final answer is.
- Have your student ID ready to be checked when submitting your exam.

DO NOT WRITE ABOVE THIS LINE!!

SECTION 1: MULTIPLE CHOICE

For each multiple choice question below (Problems 1-6), show all necessary work in addition to circling the correct response. Each question will be graded in terms of both your circled choice and the work provided. In order to earn full credit, you must have a clearly circled answer and written work to support that answer.

1. (10 points) Evaluate the following integral: $\displaystyle\int_0^{\frac{\pi}{2}} x \sin x \, dx$

   (a) $\frac{1}{3}$

   (b) $\frac{1}{2}$

   (c) $\frac{2}{3}$

   (d) $\frac{3}{4}$

   $\boxed{\text{(e) } 1}$

$$\int_0^{\pi/2} x \sin x \, dx$$

$$\left[-x\cos x + \sin x\right]_0^{\pi/2}$$

$$\left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - (0+0)$$

$$(0+1) - (0+0)$$

$$1$$

$\int x \sin x \, dx$

$u = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = -\cos x$

$\int u\,dv = uv - \int v\,du$

$\int x \sin x \, dx = -x\cos x - \int -\cos x \, dx$

$= -x\cos x + \int \cos x \, dx$

$= -x\cos x + \sin x + C$

2. (9 points) The horizontal base of a cylindrical tank is a circular disk of radius 1 m (meter) and its height is 4 m. It is half filled with a liquid that weighs 3 N/m^3 = (density 0.3 kg/m^3)(gravity 10 m/s^2). How much work, in Nm, is needed to pump out that liquid over the top rim?

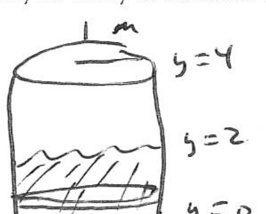
   (a) 6π

   (b) 8π

   (c) 12π

   (d) 15π

   (e) 18π



$A_0 = \pi r^2$

$A_0 = \pi \cdot 1^2 = \pi$

$V_{Disk} = \pi \Delta y \ m^3$

Mass Disk $= 0.3 \frac{kg}{m^3} \cdot \pi \Delta y \ m^3 = 0.3\pi\Delta y$ kg

$F = ma = 0.3\pi\Delta y \ kg \cdot 10 \frac{m}{s^2} = 3\pi\Delta y \ N$

Work $=$ F × dist $= 3\pi\Delta y\,[4-y]$

$$W = \int_0^2 3\pi(4-y)\,dy = 3\pi\left[4y - \frac{y^2}{2}\right]_0^2$$

$$= 3\pi\left[\left(8 - \frac{(2)^2}{2}\right) - (0-0)\right] = 3\pi[8-2]$$

$$18\pi$$

3. (10 points) Evaluate the following integral: $\int_0^{\frac{\pi}{2}} \sin^3 x\, dx$

(a) 0

(b) $\frac{1}{3}$

(c) $\boxed{\frac{2}{3}}$

(d) 1

(e) $\frac{4}{3}$

$\int \sin^3 x\, dx =$

$\int \sin^2 x \sin x\, dx =$

$\int (1-\cos^2 x)\sin x\, dx$

$u = \cos x \quad \frac{du}{dx} = -\sin x$

$\frac{du}{-\sin x} = dx$

$\int (1-u^2)\sin x \frac{du}{-\sin x}$

$-\int (1-u^2)\, du$

$-\left[u - \frac{u^3}{3}\right] + C$

$-u + \frac{u^3}{3} + C$

$-\cos x + \frac{\cos^3 x}{3} + C$

$\int_0^{\pi/2} \sin^3 x\, dx = \quad \pi/2$

$-\cos x + \frac{\cos^3 x}{3}\Big]_{x=0}$

$\left[-\cos\frac{\pi}{2} + \frac{(\cos\frac{\pi}{2})^3}{3}\right] - \left[-\cos 0 + \frac{(\cos 0)^3}{3}\right]$

$(0+0) - (-1+\frac{1}{3}) = \frac{2}{3}$

4. (6 points) The region bounded by the graph $y = 1 + x^2$ and the line $y = 2$ is rotated about the x-axis to form a solid. (For supporting work to earn full credit, sketch a graph of the enclosed region and label intersection points.) The volume of the solid is given by:

(a) $\int_1^2 2\pi\left[2^2 - (1+x^2)^2\right] dx$

(b) $\int_1^2 \pi\left[2^2 - (1+x^2)^2\right] dx$

(c) $\int_{-1}^1 2\pi\left[2^2 - (1+x^2)^2\right] dx$

(d) $\int_{-1}^1 \pi\left[2^2 - (1+x^2)^2\right] dx$

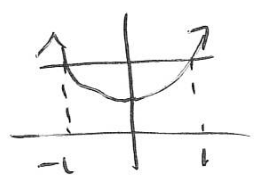
(e) $\int_1^2 \pi\left(1 - x^2\right)^2 dx$

$1 + x^2 = 2$

$x^2 - 1 = 0$

$(x-1)(x+1) = 0$

$x = -1 \text{ or } x = 1$



$R = 2$

$r = 1 + x^2$

$A = \pi R^2 - \pi r^2$

$A = \pi(2)^2 - \pi(1+x^2)^2$

$V_{washer} = \left[\pi(2)^2 - \pi(1+x^2)^2\right]\Delta x$

$V = \int_{-1}^{1} \left[\pi(2)^2 - \pi(1+x^2)^2\right] dx$

$V = \int_{-1}^{1} \pi\left[2^2 - (1+x^2)^2\right] dx$

DO NOT WRITE ABOVE THIS LINE!!

5. (10 points) Given $f(x) = \dfrac{1}{1-x^2}$, use Taylor/Maclaurin series to compute $f^{(4)}(0)$, the fourth derivative of $f(x)$ evaluated at $x = 0$.

(a) 0

(b) 4

(c) 8

(d) 12

(e) 24

$$f(x) = \frac{1}{1-x^2} = \frac{a_0}{1-r}$$

$$a_0 = 1, \quad r = x^2$$

$$f(x) = 1 + x^2 + x^4 + x^6 + x^8 + \cdots$$

$$f'(x) = 2x + 4x^3 + 6x^5 + 8x^7 + \cdots$$

$$f''(x) = 2 + 12x^2 + 30x^4 + 56x^6 + \cdots$$

$$f'''(x) = 24x + 120x^3 + 336x^5 + \cdots$$

$$f''''(x) = 24 + 360x^2 + 1680x^4 + \cdots$$

$$f''''(0) = 24 + 360(0)^2 + 1680(0)^4 + \cdots$$

$$= 24$$

DO NOT WRITE ABOVE THIS LINE!!

6. (12 points) Find the interval of convergence for the following series: $\sum_{k=1}^{\infty} \frac{2^k}{k+1} x^{2k}$

(a) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(b) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(c) $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(d) $\left(-\frac{1}{2}, \frac{1}{2}\right)$

(e) $\left(-\frac{1}{2}, \frac{1}{2}\right]$

Proceed using the ~~integral~~ ratio test

$|a_k| = \frac{2^k}{k+1} |x|^{2k}$ $\quad |a_{k+1}| = \frac{2^{k+1}}{k+1+1} |x|^{2(k+1)}$

$|a_{k+1}| = \frac{2^{k+1}}{k+2} |x|^{2k+2}$

$$\frac{|a_{k+1}|}{|a_k|} = \frac{2^{k+1} |x|^{2k+2}}{k+2} \div \frac{2^k}{k+1} |x|^{2k}$$

$$\frac{|a_{k+1}|}{|a_k|} = \frac{2^{k+1} |x|^{2k+2}}{k+2} \cdot \frac{k+1}{2^k |x|^{2k}} = \frac{2(k+1)}{(k+2)} x^2$$

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{2(k+1)}{(k+2)} x^2 = 2x^2 \lim_{k\to\infty} \frac{(k+1)}{(k+2)} = 2x^2 \cdot 1 = 2x^2$$

The series converges when the limit is less than one.

$2x^2 < 1$

$x^2 < \frac{1}{2}$

$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Check the end pts

$f(x) = \sum_{k=1}^{\infty} \frac{2^k}{k+1} x^{2k}$

$f\left[\frac{-1}{\sqrt{2}}\right] = \sum \frac{2^k}{k+1} \left[\frac{-1}{\sqrt{2}}\right]^{2k} = \sum \frac{2^k}{k+1} \cdot \frac{1}{2^k} = \sum \frac{1}{k+1}$

Diverges

$f\left[\frac{1}{\sqrt{2}}\right] = \sum \frac{2^k}{k+1} \left[\frac{1}{\sqrt{2}}\right]^{2k} = \sum \frac{2^k}{k+1} \cdot \frac{1}{2^k} = \sum \frac{1}{k+1}$

Diverse

Interval of convergence is $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

DO NOT WRITE ABOVE THIS LINE!!

8. (12 points) Evaluate the following integral: $\int_1^\infty \frac{1}{x^2}\,dx$. Show all work.

Improper Integral

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b$$

$$= \lim_{b\to\infty} \left[-\frac{1}{b}\right] - \left[-\frac{1}{1}\right]$$

$$= \lim_{b\to\infty} -\frac{1}{b}^{\to 0} + 1 = \boxed{1}$$

9. (10 points) Construct a Taylor polynomial of order 3 centered at $x = 1$ for $f(x) = \sqrt{x}$.
DO NOT SIMPLIFY YOUR ANSWER.

3^{rd}-order Taylor polynomial centered at $x = 1$:

$$f(1) + f'(1)\cdot(x-1) + f''(1)\cdot\frac{(x-1)^2}{2!} + f'''(1)\cdot\frac{(x-1)^3}{3!}$$

where $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}}$, $f''(x) = -\frac{1}{4}\cdot\frac{1}{x^{3/2}}$, and $f'''(x) = \frac{3}{8}\cdot\frac{1}{x^{5/2}}$. Thus, $f(1) = 1$, $f'(1) = \frac{1}{2}$, $f''(1) = -\frac{1}{4}$, $f'''(1) = \frac{3}{8}$, which gives

$$\boxed{1 + \frac{1}{2}\cdot(x-1) - \frac{1}{4}\cdot\frac{(x-1)^2}{2!} + \frac{3}{8}\cdot\frac{(x-1)^3}{3!}}$$

SECTION 2: FREE RESPONSE
For each free response question below (Problems 7-15), show all necessary work to earn full credit.

7. (16 points) Evaluate the following integrals:

(a) $\displaystyle\int \frac{x+3}{x^2+6x+1}\,dx$

u-substitution

$u = x^2+6x+1 \qquad du = (2x+6)\,dx$

$\Rightarrow \frac{1}{2}du = (x+3)\,dx$

$= \displaystyle\int \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\cdot\ln|u| + C$

$= \boxed{\frac{1}{2}\cdot\ln|x^2+6x+1| + C}$

(b) $\displaystyle\int \frac{x+8}{(x+1)(2x-5)}\,dx$

Partial Fractions

$\dfrac{x+8}{(x+1)(2x-5)} = \dfrac{A}{x+1} + \dfrac{B}{2x-5} \Rightarrow x+8 = A\cdot(2x-5) + B\cdot(x+1)$

Plugging in $x=-1$ gives $-1+8 = A\cdot(2\cdot(-1)-5) + B\cdot\cancel{(-1+1)}^{0}$

$\Rightarrow 7 = -7A \Rightarrow \boxed{A=-1}$

Plugging in $x=\frac{5}{2}$ gives $\frac{5}{2}+8 = -1\cdot\cancel{(2\cdot\frac{5}{2}-5)}^{0} + B(\frac{5}{2}+1)$

$\Rightarrow \frac{21}{2} = B\cdot\frac{7}{2} \Rightarrow \boxed{B=3}$

$= \displaystyle\int \frac{-1}{x+1} + \frac{3}{2x-5}\,dx = \boxed{-\ln|x+1| + \frac{3}{2}\ln|2x-5| + C}$

DO NOT WRITE ABOVE THIS LINE!!

10. (16 points) Determine whether each of the following series converge or diverge. Justify your answer and name the test(s) used.

8 (a) $\sum_{k=1}^{\infty} \frac{2^{2k}}{3^{k+1}} = \sum_{k=1}^{\infty} \frac{4^k}{3^{k+1}} = \sum_{k=1}^{\infty} \frac{1}{3}\left(\frac{4}{3}\right)^k$ ← geometric

$r = 4/3$ (doesn't matter what a is)

Because $r > 1$, series diverges.

OR

divergence test $\lim_{k\to\infty} \frac{4^k}{3^{k+1}} = \lim_{k\to\infty} \frac{1}{3}\left(\frac{4}{3}\right)^k \Rightarrow \infty$

because $\lim_{k\to\infty} a_k \neq 0$ the series diverges

8 (b) $\sum_{k=0}^{\infty} \frac{4k^2+7k-1}{11k^2+17}$

divergence test

set limit...

$\lim_{k\to\infty} \frac{4k^2+7k-1}{11k^2+17} = \ldots = \frac{4}{11}$

divide by k^2

L'Hopital's

Because $\lim_{k\to\infty} a_k \neq 0$, the series diverges.

DO NOT WRITE ABOVE THIS LINE!!

11. (20 points) ✓ Compare to $\frac{1}{2k}$

8 (a) Does $\sum_{k=1}^{\infty} \frac{1}{2k+5}$ converge? Justify your answer and name the test(s) used.

Because $\frac{1}{2k+5} < \frac{1}{2k}$, this direct comparison doesn't work.

$\frac{1}{2k+5} > \frac{1}{2k+5k} = \frac{1}{7k}$ for $k \geq 2$ → direct comparison to $\frac{1}{7}\left(\frac{1}{k}\right)$ ← harmonic

appropriate work - 2 pts.

OR

limit comparison: $\lim_{k\to\infty} \frac{\frac{1}{2k+5}}{\frac{1}{2k}} = \lim_{k\to\infty} \frac{2k}{2k+5} = 1$

$\underbrace{}_{\text{with or divide by } k}$

↳ diverges.

8 (b) Does $\sum_{k=1}^{\infty} \frac{(-1)^k}{2k+5}$ converge? Justify your answer and name the test(s) used.

alt. series test

* $\lim_{k\to\infty} \frac{1}{2k+5} = 0$ ✓

* $a_{k+1} < a_k$: $\underbrace{\frac{1}{2(k+1)+5} = \frac{1}{2k+7}}_{a_{k+1}} < \underbrace{\frac{1}{2k+5}}_{a_k}$

denominator larger, fraction smaller

Thus converges.

4 (c) Does $\sum_{k=1}^{\infty} \frac{(-1)^k}{2k+5}$ converge absolutely or conditionally? Briefly explain why.

Conditionally convergent, since $\sum \frac{1}{2k+5}$ diverges but

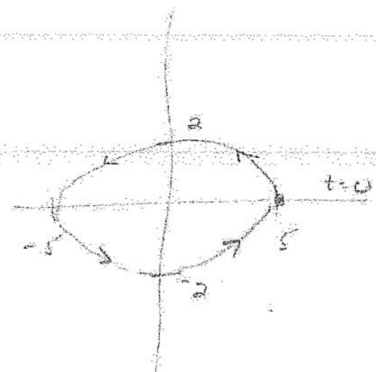
$\sum \frac{(-1)^k}{2k+5}$ converges.

DO NOT WRITE ABOVE THIS LINE!!

12. (16 points) Given the parametric equation $x = 5\cos t$ and $y = 2\sin t$ for $0 \le t \le 2\pi$:

8 (a) Sketch a graph of the curve.

- 4 pts for coordinates
- 2 pts for ellipse shape (connected)
- 2 pts for direction (counterclockwise)



8 (b) Find the slope of the tangent line at $t = \frac{\pi}{6}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{dt}} = \frac{2\cos t}{-5\sin t} \Rightarrow \frac{2\cos(\frac{\pi}{6})}{-5\sin(\pi/6)} = \frac{2(\frac{\sqrt{3}}{2})}{-5(\frac{1}{2})} = -\frac{2\sqrt{3}}{5}$$

@ $t = \pi/6$

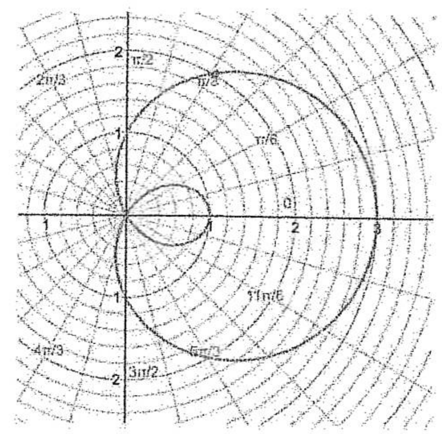
DO NOT WRITE ABOVE THIS LINE!!

13. (22 points) On the right is a graph of $r = 1 + 2\cos\theta$:

6 (a) Translate the point on the curve at $\theta = \frac{\pi}{3}$ into Cartesian coordinates.

first find $r = 1+2\cos(\pi/3) = 1+2(\frac{1}{2}) = 2$

(also on graph). $\rightarrow$



$x = r\cos\theta = 2\cos(\pi/3)$

$= 2(\frac{1}{2}) = 1$

$y = r\sin\theta = 2\sin(\pi/3) = 2(\frac{\sqrt{3}}{2}) = \sqrt{3}$

thus $(2, \pi/3) \rightarrow (1, \sqrt{3})$

8 (b) Find the slope of the tangent line at $\theta = \frac{\pi}{2}$. $\quad r' = -2\sin\theta$

$$\text{slope} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{-2\sin^2\theta + (1+2\cos\theta)\cos\theta}{-2\sin\theta\cos\theta - (1+2\cos\theta)\sin\theta}$$

plug in $\theta = \pi/2$ $\quad = \frac{-2+0}{0-1} = 2$

8 (c) Set up but DO NOT SOLVE the integral to find the area of the inner loop.

$0 = r = 1+2\cos\theta$, $\cos\theta = -\frac{1}{2}$ @ $2\pi/3$, $4\pi/3$

$$\text{Area} = 2\int_{2\pi/3}^{\pi} \frac{1}{2}(1+2\cos\theta)^2\, d\theta \quad \leftarrow \text{w/ symmetry}$$

OR

$$\text{Area} = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(1+2\cos\theta)^2\, d\theta \quad \leftarrow \text{w/out symmetry}$$

DO NOT WRITE ABOVE THIS LINE!!

14. (18 points) For the system of equations

$$2x + 3y = 4$$
$$4x + y = -2$$

(a) Solve the system of equations by setting up an augmented matrix and reducing to upper triangular form.

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left[\begin{array}{cc|c} 2 & 3 & 4 \\ 4 & 1 & 2 \end{array}\right]$$

$$\begin{array}{c} \rightsquigarrow R_1 \\ R_2 - 2R_1 \end{array} \left[\begin{array}{cc|c} 2 & 3 & 4 \\ 0 & -5 & -10 \end{array}\right]$$

$\Rightarrow -5y = -10 \Rightarrow \boxed{y=2}$

$2x+3y=4 \Rightarrow 2x+6=4$

$\Rightarrow 2x=-2$

$\Rightarrow \boxed{x=-1}$

(b) Set up the system of equations as $A\mathbf{x} = \mathbf{b}$. Find the inverse matrix A^{-1}. Then use the inverse matrix to solve the system of equations.

$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ $\quad$ $\det A = -10$ $\Rightarrow A^{-1} = -\frac{1}{10} \cdot \begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} A x = A^{-1} b = -\frac{1}{10} \begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

15. (13 points) Given the matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$:

(a) Determine which of the following are eigenvectors of A.

i. $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $Av_1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$ eigenvector with $\lambda = 3$

ii. $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $Av_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \neq \lambda v_2$

iii. $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $Av_3 = \begin{bmatrix} -1 \\ -7 \end{bmatrix} \neq \lambda v_3$

iv. $\mathbf{v}_4 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $Av_4 = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ eigenvector with $\lambda = 2$

v. $\mathbf{v}_5 = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$ $Av_5 = \begin{bmatrix} 12 \\ -12 \end{bmatrix}$ eigenvector with $\lambda = 3$

(b) For each vector in part (a) that is an eigenvector, specify the appropriate eigenvalue.

v_1 & v_5 are eigenvectors with eigenvalue $\lambda = 3$,
and v_4 is an eigenvector with eigenvalue $\lambda = 2$.