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MATH 181 Midterm 2
November 1, 2017

Directions. Fill in each of the boxes below. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR INSTRUCTOR. Good luck!

- Circle your instructor: Jones Kashcheyeva Shulman
- VERY IMPORTANT!!! CHECK THAT THE NUMBER AT THE TOP OF EACH PAGE OF YOUR EXAM IS THE SAME. IT IS THE NUMBER PRECEDED BY A POUND (#) SIGN. IF THEY ARE NOT ALL THE SAME, NOTIFY YOUR INSTRUCTOR OR TA RIGHT AWAY.
- All of your work must fit within the boxes on each page for each question. Nothing outside of the box will be graded! If you write outside of the box, there is a good chance that your exam will not be read and therefore not graded.
- You must show all of your work.
- A solution for one problem may not go on another page.
- If you are asked to calculate an integral, make sure you justify your answer if it converges or diverges.

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1. (14 points) Fill in each of the blanks using the answers provided. Write your answer in the blank except in part (d) where you are asked to circle your answer.

(a) Given a series $\sum_{k=1}^{\infty} a_k$, if $\lim_{k\to\infty} a_k = 0$, then the Divergence Test is inconclusive.

- concludes convergence
- concludes divergence
- is inconclusive

(b) For the series $\sum_{k=1}^{\infty} b_k$ where $b_k \geq 0$, if we were to use the Ratio Test from class, then we compute

the limit $L =$ $\lim_{k\to\infty} \frac{b_{k+1}}{b_k}$ and if $L > 1$, then the series is divergent.

- $\lim_{k\to\infty} b_k$
- $\lim_{k\to\infty} |b_k|$
- $\lim_{k\to\infty} \frac{b_k}{b_{k+1}}$
- $\lim_{k\to\infty} \frac{b_{k+1}}{b_k}$

- divergent
- convergent
- unknown since the test is inconclusive

(c) The p-series test is used on series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$, and

if $p>1$ then the series is convergent. Otherwise it is divergent.

- $\sum_{k=1}^{\infty} \frac{1}{k^p}$
- $\sum_{k=1}^{\infty} \frac{1}{p^k}$
- $\sum_{k=1}^{\infty} \frac{1}{p^p}$
- $\sum_{k=1}^{\infty} \frac{p^k}{k^p}$

- $p < 1$
- $p > 1$
- $p \leq 1$
- $p \geq 1$

(d) Assume we have two series $\sum a_k$ and $\sum b_k$, where $0 \leq a_k \leq b_k$. Circle the two correct statements regarding the Comparison Test.

- If $\sum a_k$ converges, then $\sum b_k$ converges too.
- If $\sum b_k$ converges, then $\sum a_k$ converges too.
- If $\sum a_k$ diverges, then $\sum b_k$ diverges too.
- If $\sum b_k$ diverges, then $\sum a_k$ diverges too.

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2. (11 points) Consider the power series $\sum_{k=1}^{\infty} \frac{(x-4)^k}{(k+1)2^k}$.

(a) What is the center of this power series?

$x = 4$

(b) Find the interval of convergence. Remember to test the endpoints and justify your interval.

$$\lim_{k\to\infty} \left| \frac{b_{k+1}}{b_k} \right| = \lim_{k\to\infty} \left| \frac{(x-4)^{k+1}}{(k+2)2^{k+1}} \cdot \frac{(k+1)2^k}{(x-4)^k} \right|$$

$$= \lim_{k\to\infty} \frac{k+1}{k+2} \cdot \frac{1}{2} \cdot |x-4|$$

$$= \frac{1}{2}|x-4| < 1$$

$$2 < x < 6$$

endpoints

$x=2$ $\quad \sum_{k=1}^{\infty} \frac{(-2)^k}{(k+1)2^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k+1}$ converges, alternating harmonic series.

$x=6$ $\quad \sum \frac{\cancel{2^k}}{(k+1)\cancel{2^k}} = \sum_{k=1}^{\infty} \frac{1}{k+1}$ harmonic series, diverges

$\Rightarrow I = [2, 6) \qquad R = 2$

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3. (15 points) For parts (a)-(c), write your series using sigma (summation) notation. For each series you come up with, justify your answer using series tests.

(a) Give an example of a series that is divergent.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$$ $p=\frac{1}{2}$, p-test, series diverges

(b) Give an example of a series that is conditionally convergent.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$ alternating harmonic series is convergent

but $\sum_{k=1}^{\infty} \frac{1}{k}$ harmonic series is divergent.

$\Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ conditionally convergent.

(c) Give an example of a series that is absolutely convergent.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \rightsquigarrow \sum_{k=1}^{\infty} \left| \frac{(-1)^k}{k^2} \right| = \sum \frac{1}{k^2}$$ $p=2$, p-test convergent

$\Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ is absolutely convergent

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4. (10 points) Determine if the following series is convergent or divergent. $\sum_{n=1}^{\infty} \frac{2^n n^5}{n!}$

Ratio Test

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}(n+1)^5}{(n+1)!} \cdot \frac{n!}{2^n \cdot n^5}\right|$$

$$= \lim_{n\to\infty} \left|\frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} \cdot \underbrace{\left(\frac{n+1}{n}\right)^5}_{\to 1}\right|$$

$$= \lim_{n\to\infty} 2 \cdot \frac{1}{n+1}$$

$$= 0$$

by ratio test this series is convergent!

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5. (10 points) Find the sum of the following series. $\sum_{m=1}^{\infty} \frac{(-1)^m + 3}{5^m} = S$

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{5^m} = \sum_{m=1}^{\infty} \left(-\frac{1}{5}\right)^m$$

geometric series

$a = -\frac{1}{5}$ $r = -\frac{1}{5}$

$$= \frac{-\frac{1}{5}}{1+\frac{1}{5}} = \frac{-\frac{1}{5}}{\frac{6}{5}}$$

$$= -\frac{1}{\cancel{5}} \cdot \frac{\cancel{5}}{6} = -\frac{1}{6}$$

$$\sum_{m=1}^{\infty} \frac{3}{5^m}$$ $a = \frac{3}{5}$ $r = \frac{1}{5}$

$$= \frac{\frac{3}{5}}{1-\frac{1}{5}} = \frac{\frac{3}{5}}{\frac{4}{5}}$$

$$= \frac{3}{5} \cdot \frac{5}{4} = \frac{3}{4}$$

$$\Rightarrow S = -\frac{1}{6} + \frac{3}{4}$$

$$= \frac{-4+18}{24} = \frac{14}{24} = \boxed{\frac{7}{12}}$$

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6. (10 points) Find the limit of the sequence $\left\{\dfrac{n^3+2n-5}{e^{3n}}\right\}$ or show that it is divergent.

$$= \lim_{x\to\infty} \frac{x^3+2x-5}{e^{3x}} \quad \text{of type } \frac{\infty}{\infty} \Rightarrow \text{use L'Hôpital}$$

$$= \lim_{x\to\infty} \frac{3x^2+2}{3e^{3x}} \quad \frac{\infty}{\infty}, \text{ use L'Hôpital}$$

$$= \lim_{x\to\infty} \frac{6x}{9e^{3x}} \quad \frac{\infty}{\infty}, \text{ use L'H}$$

$$= \lim_{x\to\infty} \frac{6}{9e^{3x}} = \boxed{0}$$

7. (10 points) Consider the series $\sum_{n=2}^{\infty} \frac{1}{n^n} = \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5}$. Find a p-series to compare to the given one to determine convergence or divergence. [Note: You must use the comparison test here.]

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges $p=2$ p-test | $\frac{1}{n^2} > \frac{1}{n^4} > 0$

by comparison since $\sum \frac{1}{n^2}$ converges,

so does $\sum_{n=2}^{\infty} \frac{1}{n^4}$

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8. (10 points) Consider the function $f(x) = \ln(1 + x)$.

(a) Find the third order Taylor polynomial of f centered at 0.

$f(x) = \ln(1+x)$ $\qquad f(0) = \ln(1) = 0$

$f'(x) = \frac{1}{1+x}$ $\qquad f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2}$ $\qquad f''(0) = -1$

$f'''(x) = \frac{+2}{(1+x)^3}$ $\qquad f'''(0) = 2$

$\Rightarrow T_3(x) = 0 + x - \frac{1}{2}x^2 + \frac{2}{3!}x^3$

$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

(b) Using your answer from part (a), estimate $\ln(2)$. Write your answer as a reduced fraction. [For reference, the exact value of $\ln(2)$ to four decimals is 0.6931.]

$\ln 2 \approx T_3(2) = 2 - \frac{1}{2}\cdot 2^2 + \frac{1}{3}\cdot 2^3$

$= 2 - 2 + \frac{8}{3}$

$= \boxed{\frac{8}{3}}$

9. (10 points) Consider the sum $\displaystyle\sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{n+1} - \frac{1}{n+3}\right)}_{a_n}$.

(a) Find S_6, the sixth term in the sequence of partial sums, by filling in the denominators

$$S_6 = \underbrace{\left(\frac{1}{2} - \cancel{\frac{1}{4}}\right)}_{a_1} + \underbrace{\left(\frac{1}{3} - \cancel{\frac{1}{5}}\right)}_{a_2} + \underbrace{\left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right)}_{a_3} + \underbrace{\left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right)}_{a_4} + \underbrace{\left(\cancel{\frac{1}{6}} - \frac{1}{8}\right)}_{a_5} + \underbrace{\left(\cancel{\frac{1}{7}} - \frac{1}{9}\right)}_{a_6}$$

(b) Notice that this series telescopes. Simplify S_6 by cancelling terms (don't worry about combining fractions).

$$S_6 = \frac{1}{2} + \frac{1}{3} - \frac{1}{8} - \frac{1}{9}$$

(c) Using parts (a) and (b), find a formula for S_n, the nth term in the sequence of partial sums.

$$S_n = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$

(d) Using part (c), find the sum of the series. Explain your answer.

$$\lim S_n = \frac{1}{2} + \frac{1}{3} - 0 - 0 = \frac{2+3}{6} = \frac{5}{6}$$