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1. **(10 pt)** Consider the three points $P(1,2,3)$, $Q(3,0,4)$ and $R(4,-2,3)$.
   (a) Compute $\overrightarrow{PQ} \cdot \overrightarrow{PR}$ and $\overrightarrow{PQ} \times \overrightarrow{PR}$.
   (b) Find the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Leave your answer in exact form.
   (c) Find the area of the triangle PQR.

a) $\overrightarrow{PQ} = \langle 3-1, 0-2, 4-3 \rangle = \langle 2, -2, 1 \rangle$

$\overrightarrow{PR} = \langle 4-1, -2-2, 3-3 \rangle = \langle 3, -4, 0 \rangle$

$\overrightarrow{PQ} \cdot \overrightarrow{PR} = \langle 2, -2, 1 \rangle \cdot \langle 3, -4, 0 \rangle = 6 + 8 = 14$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 1 \\ 3 & -4 & 0 \end{vmatrix} = \vec{i}(0+4) - \vec{j}(0-3) + \vec{k}(-8+6) = \langle 4, 3, -2 \rangle$$

b) $|\overrightarrow{PQ}| = \sqrt{4+4+1} = \sqrt{9} = 3$ $\quad |\overrightarrow{PR}| = \sqrt{9+16} = \sqrt{25} = 5$

$$\cos\theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|} = \frac{14}{3 \cdot 5} = \frac{14}{15}$$

$$\theta = \cos^{-1} \frac{14}{15}$$

c) $|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{16+9+4} = \sqrt{29}$

$$\text{area} = \frac{1}{2}\sqrt{29}$$

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2. **(10 pt)** Let C be the curve, given by the vector equation

$$\mathbf{r}(t) = \langle 2t, 3\cos t, 3\sin t \rangle, 0 \le t \le \pi.$$

(a) Find $\mathbf{r}'(t)$, $\mathbf{r}''(t)$ and $|\mathbf{r}'(t)|$.

(b) Find the unit tangent vector to the curve C at $t = \frac{\pi}{4}$.

a) $\vec{r}'(t) = \langle 2, -3\sin t, 3\cos t \rangle$

$\vec{r}''(t) = \langle 0, -3\cos t, -3\sin t \rangle$

$|\vec{r}'(t)| = \sqrt{4 + 9(\sin^2 t + \cos^2 t)} = \sqrt{13}$

b) $\vec{r}'\left(\frac{\pi}{4}\right) = \langle 2, -3\sin\frac{\pi}{4}, 3\cos\frac{\pi}{4} \rangle = \langle 2, -\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \rangle$

$\vec{u}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{13}} \langle 2, -\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \rangle$

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3. **(10 pt)** Consider the function

$$f(x, y) = 4x^2y - 4x^2 + 8x - 16y + 1.$$

Find all the critical points of f and classify them as local maxima, local minima or saddle points.

$f_x = 8xy - 8x + 8 = 0$

$f_y = 4x^2 - 16 = 0 \qquad x^2 = \frac{16}{4} = 4, \quad x = \pm 2$

$x = -2: \quad -16y + 16 + 8 = 0$

$16y = 24$

$y = \frac{24}{16} = \frac{3}{2}, \qquad \left(-2, \frac{3}{2}\right)$

$x = 2: \quad 16y - 16 + 8 = 0$

$16y = 8$

$y = \frac{1}{2}, \qquad \left(2, \frac{1}{2}\right)$

$f_{xx} = 8y - 8 \qquad f_{xy} = 8x \qquad f_{yy} = 0$

$D(x,y) = (8y-8)\cdot 0 - 64x^2 = -64x^2$

$D\left(2, \frac{1}{2}\right) = -64 \cdot 4 < 0$ saddle point

$D\left(-2, \frac{3}{2}\right) = -64 \cdot 4 < 0$ saddle point

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4. **(10pt)** Use the method of Lagrange multipliers to find the maximum and minimum values of

$$f(x, y, z) = x + 2y - z$$

subject to $x^2 + y^2 + z^2 = 1$.

$\nabla f = \langle 1, 2, -1 \rangle \qquad \nabla g = \langle 2x, 2y, 2z \rangle$

$1 = 2\lambda x \qquad x = \frac{1}{2\lambda}$

$2 = 2\lambda y \qquad y = \frac{2}{2\lambda} \qquad \frac{1}{4\lambda^2} + \frac{4}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$

$-1 = 2\lambda z \qquad z = -\frac{1}{2\lambda}$

$\frac{6}{4\lambda^2} = 1, \; \lambda = \pm\sqrt{\frac{3}{2}}$

$\lambda = -\sqrt{\frac{3}{2}}$: $\quad x = -\frac{1}{2\sqrt{3/2}} = -\frac{1}{\sqrt{6}} \quad y = -\frac{2}{2\sqrt{3/2}} = -\frac{2}{\sqrt{6}},$

$z = \frac{1}{\sqrt{6}}, \quad \left(-\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$\lambda = \sqrt{\frac{3}{2}}$: $\quad x = \frac{1}{\sqrt{6}}, \; y = \frac{2}{\sqrt{6}}, \; z = -\frac{1}{\sqrt{6}},$

$\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$

$f\left(-\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = -\frac{1}{\sqrt{6}} - \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} = -\frac{4}{\sqrt{6}}$ min

$f\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) = \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} = \frac{4}{\sqrt{6}}$ max

5. **(10 pt)** For the integral

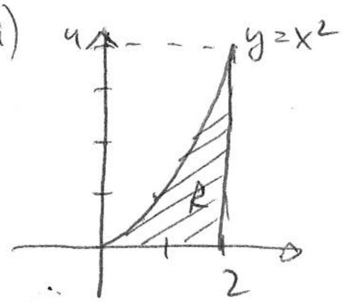
$$\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy$$

(a) Sketch the region of integration.

(b) Reverse the order of integration.

(c) Evaluate the integral from (b).

a)



$R = \{(x,y) \mid 0 \le y \le 4, \sqrt{y} \le x \le 2\}$

$x = \sqrt{y}$

$y = x^2$

b) $R = \{(x,y) \mid 0 \le x \le 2, 0 \le y \le x^2\}$

$\int_0^2 \int_0^{x^2} e^{x^3}\, dy\, dx = \int_0^2 e^{x^3} y \Big|_0^{x^2} dx =$

$\int_0^2 x^2 e^{x^3}\, dx$

$u = x^3, \quad du = 3x^2\, dx$

if $x = 0$ then $u = 0$

$x = 2$ $\quad u = 8$

$\int_0^2 x^2 e^{x^3}\, dx = \frac{1}{3} \int_0^8 e^u\, du = \frac{1}{3} e^u \Big|_0^8 =$

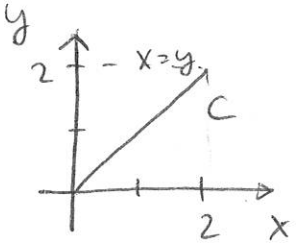
$\frac{1}{3}(e^8 - e^0) = \frac{1}{3}(e^8 - 1)$

6. **(10pt)** Let C be a line segment from $(0,0)$ to $(2,2)$, and let $f(x,y) = x^2 + y$.

   (a) Write down a vector equation $\mathbf{r}(t)$ of the line segment, that is, find a parametrization of C.

   (b) Compute $\mathbf{r}'(t)$ and $|\mathbf{r}'(t)|$.

   (c) Compute the scalar line integral $\int_C f ds$.



$t = x$, so $\vec{r}(t) = \langle t, t \rangle$, $0 \le t \le 2$.

$\vec{r}\,'(t) = \langle 1, 1 \rangle$

$|\vec{r}\,'(t)| = \sqrt{1+1} = \sqrt{2}$

$$\int_C f ds = \int_0^2 (t^2 + t)\sqrt{2}\, dt = \sqrt{2}\left(\frac{t^3}{3} + \frac{t^2}{2}\right)\Big|_0^2 =$$

$$\sqrt{2}\left(\frac{8}{3} + \frac{4}{2}\right) = \sqrt{2}\,\frac{16+12}{6} = \sqrt{2}\,\frac{28}{6} = \frac{14\sqrt{2}}{3}$$

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7. **(10 pt)** Consider a conservative vector field $\mathbf{F} = \langle z^3y, z^3x + 2, 3z^2xy + 1\rangle$.

(a) Find a potential function for $\mathbf{F}$.

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a path from the point $A(0,5,0)$ to $B(5,0,1)$.

a) $\varphi_x = f = z^3y, \quad \varphi_y = g = z^3x+2, \quad \varphi_z = h = 3z^2xy+1$

$\varphi = \int z^3y\,dx = z^3yx + C(y,z)$

$\varphi_y = z^3x + C_y(y,z) = z^3x+2$, so $C_y(y,z) = 2$

$C(y,z) = \int 2\,dy = 2y + d(z)$

$\varphi = z^3yx + 2y + d(z)$

$\varphi_z = 3z^2yx + d'(z) = 3z^2xy+1$, so $d'(z) = 1$

$d(z) = \int 1\,dz = z + K$, take $K=0$

$\boxed{\varphi = z^3yx + 2y + z}$

b) $\int_C \vec{F}\cdot d\vec{r} = \varphi(B) - \varphi(A) = 1 - 2\cdot 5 = -9$

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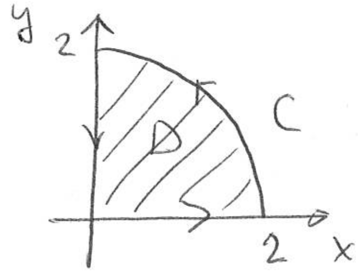
8. **(10pt)** Let D be the region enclosed by the circle $x^2 + y^2 = 4$ in the part of the plane where $x \geq 0$ and $y \geq 0$. Let C be the boundary curve of D equipped with counterclockwise orientation. Let $\mathbf{F} = \langle 2xy, x^2 \rangle$.

   (a) Sketch D and C (be sure to indicate the orientation of C).

   (b) Describe D in polar coordinates.

   (c) Using Green's Theorem, compute the outward flux $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$.

a)



b) $D = \{(r, \theta) \mid 0 \leq \theta \leq \frac{\pi}{2}, \; 0 \leq r \leq 2\}$

c) $\oint_C \vec{F} \cdot \vec{n}\, ds = \iint_R f_x + g_y \, dA =$

$\iint_R 2y \, dA = \int_0^{\frac{\pi}{2}} \int_0^2 2r \sin\theta \, r \, dr \, d\theta =$

$\int_0^{\frac{\pi}{2}} 2\sin\theta \left(\frac{r^3}{3} \Big|_0^2 \right) d\theta = \int_0^{\frac{\pi}{2}} \frac{8}{3} 2 \sin\theta \, d\theta =$

$\frac{16}{3} (-\cos\theta) \Big|_0^{\frac{\pi}{2}} = \frac{16}{3} \left(-\cos\frac{\pi}{2} + \cos 0 \right) =$

$\frac{16}{3}$

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9. **(10 pt)** Given the vector field on $\mathbb{R}^3$,

$$\mathbf{F} = \langle 3xz, 2y^2 - y, 2x + 1 \rangle.$$

(a) Compute the divergence and the curl of **F**.

(b) Determine if the vector field **F** is conservative. Justify your answer.

a) $\operatorname{div} \vec{F} = 3z + 4y - 1.$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xz & 2y^2 - y & 2x+1 \end{vmatrix} =$$

$$\vec{i}(0-0) - \vec{j}(2-3x) + \vec{k}(0-0) =$$

$$\langle 0, 3x-2, 0 \rangle$$

b) $\vec{F}$ is not conservative since $\operatorname{curl} \vec{F} \neq \vec{0}$.

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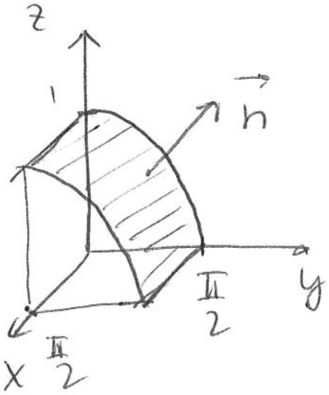
10. **(10 pt)** Compute the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where S is the surface $z = \cos y$ for $0 \le x \le \frac{\pi}{2}$ and $0 \le y \le \frac{\pi}{2}$, oriented with the upward normal, and $\mathbf{F} = \langle 1, \cos x, y^2 \rangle$.

$\vec{r}(u,v) = \langle u, v, \cos v \rangle, \quad R = \{(u,v) \mid 0 \le u, v \le \frac{\pi}{2}\}$



$\vec{F} = \langle 1, \cos u, v^2 \rangle$

$\vec{t}_u = \langle 1, 0, 0 \rangle \qquad \vec{t}_v = \langle 0, 1, -\sin v \rangle$

$$\vec{t}_u \times \vec{t}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & -\sin v \end{vmatrix} =$$

$\vec{i}(0) - \vec{j}(-\sin v) + \vec{k} \cdot 1 = \langle 0, \sin v, 1 \rangle,$
upward orientation

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \langle 1, \cos u, v^2 \rangle \cdot \langle 0, \sin v, 1 \rangle\, dv\, du =$$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (\cos u \sin v + v^2)\, dv\, du = \int_0^{\frac{\pi}{2}} -\cos u \cos v + \frac{v^3}{3}\Big|_0^{\frac{\pi}{2}} du =$$

$$\int_0^{\frac{\pi}{2}} \left(-\cos u(0 - 1) + \frac{\pi^3}{24}\right) du = \int_0^{\frac{\pi}{2}} \left(\cos u + \frac{\pi^3}{24}\right) du =$$

$$\left(\sin u + \frac{\pi^3}{24} u\right)\Big|_0^{\frac{\pi}{2}} = 1 + \frac{\pi^4}{48}$$