MATH 220 Final Exam Fall 2011

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
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<td>(20)</td>
<td>(15)</td>
<td>(15)</td>
<td>(15)</td>
<td>(15)</td>
<td>(15)</td>
<td>(15)</td>
<td>(15)</td>
<td>(15)</td>
<td>(20)</td>
</tr>
</tbody>
</table>

SCORE ______ / 200

Name (print) ______________________ Signature ______________________

Instructor (circle one) Greenblatt Knessl Nicholls

Some potentially useful formulas:

\[
\int x \sin(Ax) \, dx = -\frac{x}{A} \cos(Ax) + \frac{1}{A^2} \sin(Ax), \quad \int x \cos(Ax) \, dx = \frac{x}{A} \sin(Ax) + \frac{1}{A^2} \cos(Ax)
\]

\[
y_2(t) = y_1(t) \int^t [y_1(u)]^{-2} \exp \left( -\int^u p(v) \, dv \right) \, du
\]

\[
y_p(t) = y_1(t) \int^t -\frac{g(u) y_2(u)}{W[y_1, y_2](u)} \, du + y_2(t) \int^t \frac{g(u) y_1(u)}{W[y_1, y_2](u)} \, du, \quad W[y_1, y_2](t) = y_1(y) y'_2(t) - y'_1(t) y_2(t)
\]

\[
\mathcal{L} \{ 1 \} = \frac{1}{s}, \quad \mathcal{L} \{ e^{at} \} = \frac{1}{s-a}, \quad \mathcal{L} \{ t^n \} = \frac{n!}{s^{n+1}}, \quad \mathcal{L} \{ \sin(bt) \} = \frac{b}{s^2 + b^2}, \quad \mathcal{L} \{ \cos(bt) \} = \frac{s}{s^2 + b^2},
\]

\[
\mathcal{L} \{ e^{at} t^n \} = \frac{n!}{(s-a)^{n+1}}, \quad \mathcal{L} \{ e^{at} f(t) \} = \mathcal{L} \{ f \} (s-a) = F(s-a),
\]

\[
\mathcal{L} \{ f'(t) \} = sF(s) - f(0), \quad \mathcal{L} \{ f''(t) \} = s^2 F(s) - sf(0) - f'(0),
\]

\[
\mathcal{L} \{ f(t-a)u(t-a) \} = e^{-as} F(s), \quad \mathcal{L} \{ u(t-a) \} = \frac{e^{-as}}{s}, \quad \mathcal{L} \{ \delta(t-a) \} = e^{-as}
\]
1. (8+12) Consider the equation

\[ y'' + y = t^2 + e^t. \]

(a) Find the general solution to the homogeneous equation (with right hand side replaced by 0).

(b) Find a particular solution to the inhomogeneous equation.

2. (10+10) Solve

(a) \[ \frac{dy}{dx} = \frac{y}{x} + xe^x, \]

(b) \[ \frac{dy}{dx} = \frac{\sin x - ye^{xy}}{xe^{xy} + 1}. \]

3. (20) Use the Euler method with two steps to approximate \( y(1) \), where \( y' = (1 + x)y + x^2, \quad y(0) = 1. \)

4. (3+12) Let \( p(t) \) be the population of some species, which satisfies the ODE

\[ p' = 3p - p^2. \]

(a) If \( p(0) = 0 \) what is \( p(t) \) for \( t > 0 \)?

(b) Solve for \( p(t) \) for the initial condition \( p(0) = 1 \), and then evaluate the limit of \( p(t) \) as \( t \to \infty \).
5. (8+7) Solve

(a) \( t^2 y'' - ty' + y = 0 \), \hspace{1cm} (b) \( t^2 y'' + ty' + 4y = 0 \).

6. (3+12) Consider the ODE

\[ y'' + 2ty' - 4y = 0. \]

(a) Is \( y_1(t) = 2t^2 + 1 \) a solution?

(b) Find the general solution. (Hint: Consider the method of reduction of order. You may leave your answer as an integral.)

7. (15) Find the general solution to

\[ y'' - 2y' + y = \frac{2e^t}{t^3}. \]

8. (15) Find the general solution to the system

\[ x' = 2x + y, \hspace{1cm} y' = x + 2y. \]

9. (15) Use Laplace transforms to solve

\[ y'' + y' = e^t, \hspace{1cm} y(0) = 1, \hspace{1cm} y'(0) = 0. \]

To receive any credit, you must use Laplace transforms.
10. (15) Use Laplace transforms to solve
\[ y'' + 4y = \delta(t - 1), \quad y(0) = 0, \quad y'(0) = 1. \]
To receive any credit, you must use Laplace transforms.

11. (15) Solve the heat equation
\[ u_t = \beta u_{xx}; \quad 0 < x < 1, \quad t > 0 \]
\[ u(0, t) = 0, \quad u(1, t) = 0 \]
\[ u(x, 0) = 2 \sin(\pi x) + \sin(4\pi x). \]
Start by writing the general solution that satisfies the PDE and boundary conditions, and then find the solution that also satisfies this initial condition.

12. (3+2+3+12) Let \( f(x) = 2x \) for \( 0 < x < 1. \)

(a) Sketch the even extension of \( f(x) \) over the range \(-1 < x < 1.\)

(b) Does the Fourier series of (the extended) \( f(x) \) involve sines, cosines or both?

(c) What is the average value of (the extended) \( f(x) \) over a period?

(d) Compute explicitly the Fourier series of (the extended) \( f(x). \)