MATHEMATICS 220: FINAL EXAM
University of Illinois at Chicago
(Abramov, Awanou, Nicholls)
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Please read the exam carefully and follow all instructions. **SHOW ALL OF YOUR WORK.** Please put a box around your final answer.

1. (20 points) Solve the initial value problem

$$\frac{dy}{dx} = x^2(1+y), \qquad y(0) = 1.$$

2. (20 points) Solve the initial value problem

$$(2xy+3)\,dx + (x^2-1)\,dy = 0, \qquad y(0) = 0.$$

3. (25 points) A large 100L tank is initially filled with fresh water. At time $t = 0$, a brine solution begins to enter the tank at the rate of 5 L/min with concentration of 0.2 kg/L. The well-stirred solution is removed from the tank at the same rate of 5 L/min. Denote the amount (mass in kg) of salt in the tank as x, and then find the formula for $x(t)$ for $t \geq 0$.

4. (20 points) Find the general solution to the equation

$$y'' + 2y' + 2y = e^{-t}\cos(2t).$$

   Use the **Method of Undetermined Coefficients** to find a particular solution for the non–homogeneous equation. **(Any other method will receive no credit.)**

5. (25 points) Solve the initial value problem using differential operators

$$\begin{aligned} x' &= 4x + y, \qquad & x(0) &= 1 \\ y' &= -2x + y, \qquad & y(0) &= 0. \end{aligned}$$

6. (20 points) Compute the inverse Laplace transform of

$$F(s) = \frac{s}{s^2 - 4s + 5}.$$

7. (20 points) Using the **Method of Laplace transforms** solve the initial value problem

$$y''(t) + 14y'(t) + 58y(t) = \delta(t-8), \quad y(0) = 0, \quad y'(0) = 0.$$

(Any other method will receive no credit.)

8. (25 points) Find the values of λ for which the given problem has a nontrivial solution. Also determine the corresponding nontrivial solutions.

$$y'' + \lambda y = 0, \quad 0 < x < 1, \quad y'(0) = 0, \quad y(1) = 0.$$

9. (25 points) Consider the function

$$f(x) = 2, \quad 0 < x < \pi.$$

   (a) (19 points) Compute the **Fourier sine series** of this function.
   (b) (2 points) To what value does this series converge at $x = 0$? Why?
   (c) (2 points) To what value does this series converge at $x = \pi/2$? Why?
   (d) (2 points) To what value does this series converge at $x = \pi$? Why?

List of Laplace Transforms

1. $\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0$
2. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
3. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$
4. $\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}, \quad s > 0$
5. $\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}, \quad s > 0$
6. $\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}, \quad s > a$
7. $\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}, \quad s > a$
8. $\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}, \quad s > a$
9. $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$
10. $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
11. $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
12. $\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$
13. $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f\}(s)$
14. $\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)$
15. $\mathcal{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s}$
16. $\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$
17. If f has period T then
$$\mathcal{L}\{f\}(s) = \frac{F_T(s)}{1-e^{-sT}} = \frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$$
18. $\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$

List of PDE Formulae

1. The solution of the homogeneous heat equation $u_t = \beta^2 u_{xx}$ with Dirichlet boundary conditions is:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(\beta n\pi/L)^2 t} \sin\left(\frac{n\pi}{L}x\right).$$

2. The solution of the homogeneous heat equation $u_t = \beta^2 u_{xx}$ with Neumann boundary conditions is:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-(\beta n\pi/L)^2 t} \cos\left(\frac{n\pi}{L}x\right).$$

3. The inhomogeneous heat equation has a solution of the form $u(x,t) = v(x) + w(x,t)$, where v is the steady–state solution and w solves a homogeneous heat equation.

4. The solution of the homogeneous wave equation $u_{tt} = \alpha^2 u_{xx}$ with Dirichlet boundary conditions is:

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\alpha\frac{n\pi}{L}t\right) + b_n \sin\left(\alpha\frac{n\pi}{L}t\right) \right\} \sin\left(\frac{n\pi}{L}x\right).$$