MATHEMATICS 220: EXAM II
University of Illinois at Chicago (Professor Nicholls)
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Please read the exam carefully and follow all instructions. **SHOW ALL OF YOUR WORK.** Please put a box around your final answer.

1. (20 points) Determine the form of a particular solution for the differential equation. *Do not evaluate coefficients.*

$$y''(t) + 4y'(t) + 13y(t) = -2t^4 e^{-2t}\cos(3t).$$

2. (20 points) Consider the system of first order ODEs

$$\begin{aligned} x'(t) &= 4y(t) \\ y'(t) &= -4x(t). \end{aligned}$$

   (a) (12 points) Solve the phase plane equation for this system.

   (b) (8 points) Sketch by hand several representative trajectories (with their flow arrows).

3. (20 points) Find the general solution of

$$t^2y''(t) - ty'(t) + 17y(t) = 0.$$

4. (20 points) Find the Laplace transform of

$$f(t) = te^{3t}\sin(4t) + 3t^2e^{-t}.$$

5. (20 points) A 3 kg mass is attached to a spring hanging from the ceiling, thereby causing the spring to stretch 98 cm upon coming to rest at equilibrium. At time $t = 0$ the mass is displaced 1 m below the equilibrium position and released. At this same instant, an external force $F(t) = 25.5\ \cos(t)$ N is applied to the system. The damping constant for the system is 6 N–sec/m.

   (a) (5 points) Based upon the rest position at equilibrium find the spring constant k.

   (b) (5 points) Find the homogeneous solution.

   (c) (5 points) Find a particular solution.

   (d) (5 points) Using the initial conditions find the equation of motion for the mass.

List of Laplace Transforms

1. $\mathcal{L}\{1\} = \dfrac{1}{s}, \quad s > 0$
2. $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}, \quad s > a$
3. $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad s > 0$
4. $\mathcal{L}\{\sin(bt)\} = \dfrac{b}{s^2+b^2}, \quad s > 0$
5. $\mathcal{L}\{\cos(bt)\} = \dfrac{s}{s^2+b^2}, \quad s > 0$
6. $\mathcal{L}\{e^{at}t^n\} = \dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
7. $\mathcal{L}\{e^{at}\sin(bt)\} = \dfrac{b}{(s-a)^2+b^2}, \quad s > a$
8. $\mathcal{L}\{e^{at}\cos(bt)\} = \dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$
9. $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$
10. $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$
11. $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$
12. $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
13. $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
14. $\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$
15. $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \dfrac{d^n}{ds^n}\mathcal{L}\{f\}(s)$
16. $\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)$
17. $\mathcal{L}\{u(t-a)\}(s) = \dfrac{e^{-as}}{s}$
18. $\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$
19. If f has period T then
$$\mathcal{L}\{f\}(s) = \frac{F_T(s)}{1-e^{-sT}} = \frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$$
20. $\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$