Problem 1. (40 pts) Find the eigenvalues and corresponding eigenspaces for the matrix. Decide whether the matrix is diagonalizable or not. (Explain!)

\[
a) \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \quad b) \begin{pmatrix} 2 & 3 \\ -3 & 8 \end{pmatrix}.
\]

Problem 2. Let \( A = \begin{pmatrix} -7 & 14 & 10 \\ 1 & -2 & -1 \\ -8 & 16 & 11 \end{pmatrix} \).

It is given that \( A \) has eigenvalues \( \lambda_1 = 0 \), \( \lambda_2 = 3 \) and \( \lambda_3 = -1 \) with corresponding eigenvectors \( x_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \), \( x_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \) and \( x_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \).

a) (15 pts) Show that \( \{x_1, x_2, x_3\} \) is a basis of \( \mathbb{R}^3 \).

a) (15 pts) Write down a factorization \( A = XDX^{-1} \), where \( D \) is diagonal.

b) (15 pts) Find \( B \) such that \( B^3 = A \).

c) (15 pts) If map \( L : \mathbb{R}^3 \to \mathbb{R}^3 \) is given by \( L(x) = Ax \) for any \( x \in \mathbb{R}^3 \), show that \( L \) is a linear transformation.

d) (15 pts) What is the matrix representation of \( L \) with respect to the basis \( \{x_1, x_2, x_3\} \)?

e) (20 pts) Solve the initial value problem

\[
\begin{align*}
y_1' &= -7y_1 + 14y_2 + 10y_3 & y_1(0) &= 1 \\
y_2' &= y_1 - 2y_2 - y_3 & y_2(0) &= 2 \\
y_3' &= -8y_1 + 16y_2 + 11y_3 & y_3(0) &= 3
\end{align*}
\]

Problem 3. (10 pts) Translate the second order system

\[
\begin{align*}
y_1'' &= 2y_1 - y_2 + y_1' \\
y_2'' &= y_1 - 2y_2 + y_1'
\end{align*}
\]

into a first order system of linear differential equations. (DO NOT solve it.)
Problem 4. Let 

\[
S = \text{Span} \left\{ \begin{pmatrix} 4 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ -5 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ and } b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.
\]

a) (15 pts) Does \(b\) belong to \(S\)? Can \(S\) be equal to \(\mathbb{R}^4\)?

b) (15 pts) Find a basis of \(S\). What is the dimension of \(S\)?

c) (25 pts) Use the Gram-Schmidt process and your answer for part b) to find an orthonormal basis of \(S\).

d) (Extra credit: 20 pts) Find the projection of \(b\) onto \(S\) and the distance from \(b\) to \(S\).