Math310: Final Exam
Fall 2008

Problem 1.(40 pts) Find the eigenvalues and corresponding eigenspaces for the matrix. Decide whether the matrix is diagonalizable or not. (Explain!)

$$a)\ \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 2 & 3 \\ -3 & 8 \end{pmatrix}.$$

Problem 2. Let $A = \begin{pmatrix} -7 & 14 & 10 \\ 1 & -2 & -1 \\ -8 & 16 & 11 \end{pmatrix}$.

It is given that A has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 3$ and $\lambda_3 = -1$ with corresponding eigenvectors $x_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $x_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $x_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

a) (15 pts) Show that $\{x_1, x_2, x_3\}$ is a basis of $\mathbb{R}^3$.

a) (15 pts) Write down a factorization $A = XDX^{-1}$, where D is diagonal.

b) (15 pts) Find B such that $B^3 = A$.

c) (15 pts) If map $L : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $L(x) = Ax$ for any $x \in \mathbb{R}^3$, show that L is a linear transformation.

d) (15 pts) What is the matrix representation of L with respect to the basis $\{x_1, x_2, x_3\}$?

e) (20 pts) Solve the initial value problem

$$\begin{aligned} y_1' &= -7y_1 + 14y_2 + 10y_3 \\ y_2' &= y_1 - 2y_2 - y_3 \\ y_3' &= -8y_1 + 16y_2 + 11y_3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} y_1(0) &= 1 \\ y_2(0) &= 2 \\ y_3(0) &= 3 \end{aligned}$$

Problem 3. (10 pts) Translate the second order system

$$\begin{aligned} y_1'' &= 2y_1 - y_2 + y_2' \\ y_2'' &= y_1 - 2y_2 + y_1' \end{aligned}$$

into a first order system of linear differential equations. (DO NOT solve it.)

Problem 4. Let

$$S = \operatorname{Span}\left\{ \begin{pmatrix} 4 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ -5 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

a) (15 pts) Does b belong to S? Can S be equal to $\mathbb{R}^4$?

b) (15 pts) Find a basis of S. What is the dimension of S?

c) (25 pts) Use the Gram-Schmidt process and your answer for part b) to find an orthonormal basis of S.

d) (Extra credit: 20 pts) Find the projection of b onto S and the distance from b to S.