Problem 1. (50 pts) Find the eigenvalues and the corresponding eigenspaces for the matrix. Decide whether the matrix is diagonalizable or not. (Explain!)

\[ a) \begin{pmatrix} 2 & 4 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad b) \begin{pmatrix} 6 & 1 \\ -1 & 4 \end{pmatrix}. \]

Problem 2. Let \( A = \begin{pmatrix} -2 & 0 & 0 \\ -5 & 3 & 0 \\ -5 & 2 & 1 \end{pmatrix} \).

It is given that \( A \) has eigenvalues \( \lambda_1 = -2, \lambda_2 = 3 \) and \( \lambda_3 = 1 \) with corresponding eigenvectors \( u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \) and \( u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \).

a) (15 pts) Write down a factorization \( A = XDX^{-1} \), where \( D \) is diagonal.

b) (10 pts) Find \( A^5 \).

c) (10 pts) Find \( B \) such that \( B^3 = A \).

d) (10 pts) If the map \( L : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is given by \( L(x) = Ax \) for any \( x \in \mathbb{R}^3 \), show that \( L \) is a linear transformation.

e) (10 pts) What is the matrix representation of \( L \) with respect to the basis \( \{u_1, u_2, u_3\} \)?

f) (20 pts) What is the matrix representation of \( L \) with respect to the basis \( \{v_1, v_2, v_3\} \), where \( v_1 = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} \).
Problem 3. Let

\[ S = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} -4 \\ 6 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} \right\} \text{ and } b = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}. \]

a) (15 pts) Does \( b \) belong to \( S \)? Can \( S \) be equal to \( \mathbb{R}^4 \)?

b) (15 pts) Find a basis of \( S \). What is the dimension of \( S \)?

c) (25 pts) Use the Gram-Schmidt process to find an orthonormal basis of \( S \).

d) (10 pts) Find the projection of \( b \) onto \( S \).

e) (10 pts) Find the distance from \( b \) to \( S \).