1. The 2nd-order formula for the second derivative of a function \( u(x) \) with discretization step \( h \) is given by
\[
\frac{u''(x)}{h^2} = \frac{u(x + h) - 2u(x) + u(x - h)}{h^2} + O(h^2).
\]
Use Richardson’s extrapolation to find the 4th-order centered-difference formula.

**Answer:** One can check that
\[
\frac{u''(x)}{h^2} = \frac{f^{(4)}(x)}{12} h^2 + O(h^4) - \frac{f^{(4)}(x)}{12} h^2 + O(h^4),
\]
where \( A_2(h) \) is the approximation of order 2. Then, according to Richardson’s formula, the 4th order approximation is found as
\[
A_4(h) = \frac{2^2 A(h/2) - A(h)}{2^2 - 1},
\]
resulting in
\[
\frac{u''(x)}{h^2} \approx -\frac{1}{3h^2} u(x + h) + \frac{16}{3h^2} u(x + h/2) - \frac{10}{h^2} u(x) + \frac{16}{3h^2} u(x - h/2) - \frac{1}{3h^2} u(x - h).
\]
Replacing \( h \) with \( 2h \), we obtain
\[
\frac{u''(x)}{h^2} \approx -\frac{1}{12h^2} u(x + 2h) + \frac{5}{3h^2} u(x + h) + \frac{4}{2h^2} u(x) + \frac{4}{3h^2} u(x - h) - \frac{1}{12h^2} u(x - 2h).
\]

2. Consider the function
\[
f(x) = x \left(x - \frac{1}{2}\right)^2.
\]
Obviously, it has two roots: \( x_1 = 0 \) and \( x_2 = 1/2 \). Without actually running the calculation, answer the following questions:

a) Which root-finding method will converge faster to \( x_1 \), the Bisection method or the Newton’s method? Why?

b) Which root-finding method will converge faster to \( x_2 \), the Bisection method or the Newton’s method? Why?

**Answer:**

a) Root \( x_1 = 0 \) is of multiplicity 1, and thus the Newton’s method has second order convergence to \( x_1 = 0 \). On the other hand, the Bisection method is of first-order convergence. Therefore, the Newton method will converge faster to \( x_1 = 0 \).
b) Root \( x_2 = 1/2 \) has multiplicity 2, which results in Newton method becoming the first order (same as bisection method). However, for the bisection method to work, \( f(x) \) should have opposite signs for \( x \) smaller or greater than \( x_2 \). Therefore, the bisection method will not converge to \( x_2 \) at all.

3. Assume that the explicit trapezoid method, given by

\[
x_{n+1} = x_n + \frac{h}{2} [f(x_n) + f(x_n + hf(x_n))],
\]

is used to solve the ordinary differential equation \( x'(t) = -\alpha x(t) \), where \( \alpha > 0 \) is a constant parameter. Find the range of values for the discretization step \( h \), for which the method retains stability (that is, \( x_n \) remains bounded as \( n \to \infty \)).

**Answer:** For the equation \( x'(t) = -\alpha x(t) \) the midpoint method produces the map

\[
x_{n+1} = x_n + \frac{h}{2} [-\alpha x_n - \alpha(x_n - h\alpha x_n)] = \left(1 - h\alpha + \frac{h^2\alpha^2}{2}\right) x_n.
\]

Therefore, \( x_n = (1 - h\alpha + h^2\alpha^2/2)^n x_0 \), and for stability we need \( |1 - z + z^2/2| < 1 \), where \( z = h\alpha \). Now, the graph of \( g(z) = z^2/2 - z + 1 \) is never negative, and all we need to find are the roots of \( z^2/2 - z + 1 \), which are \( z = 0 \) and \( z = 2 \). Thus, \( g(z) = z^2/2 - z + 1 \) is less than 1 for \( 0 < z < 2 \), and therefore for stability we must have \( 0 < h\alpha < 2 \), or \( h < 2/\alpha \).
\[ f(x) = x - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{1}{7 \cdot 5!} x^7 + \ldots \]

\[ (e^{x^2} f)' = x e^{x^2} f + e^{x^2} \left[ 1 - x^2 + \frac{x^4}{3!} - \frac{x^6}{5!} + \frac{x^8}{7 \cdot 5!} \right] \]

\[ = e^{x^2} \]

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**Complex**

1. \[ f(z) = \exp\left(\frac{i}{z-\pi}\right) \frac{1}{\sin^2(\pi z)} \frac{1}{z^{2010}} \]

   singular points at \( z = 0, \pm 1, \pm 2, \pm 3, \ldots \)
   
   \( z = 0 \) is a pole of order \( 2010 + 2 = 2012 \)
   
   \( z = \pi \) is an essential singularity
   
   \( z = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \ldots \)
   
   are double poles

2. \[ z^{2012} + 2012 z + 1 = 0 \]

   how many roots inside \( |z| = 1 \)
   inside annulus \( 1 < |z| < 2 \)

   \[ |z| = 1 \] 2012 \( z \) dominates so 1 root by Rouché

   \[ |z| = 2 \] 2012 \( z \) so 2012 roots inside \( |z| = 2 \)
   
   2011 \( z \) annulus \( |z| < (1, 2) \)
\[ \text{Integral} = \frac{\pi}{2} \]
\[ \Delta u = 0 \quad \frac{\pi}{2} < r < \frac{3\pi}{2}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \]

\[ u \bigg|_{r=\frac{\pi}{2}} = \frac{1}{2} \cos \theta \quad u \bigg|_{r=\frac{\pi}{3}} = \sin(2\theta) \]

\[ u = 0 \quad \theta = \pm \frac{\pi}{2} \]

\[ U = A(r) \cos \theta + B(r) \sin(2\theta) \]

\[ = \left( \alpha r + \beta \right) \cos \theta + \left( \delta r^2 + \frac{\ell}{r^3} \right) \sin(2\theta) \]

\[ \alpha + \beta = 0 \quad \delta + \ell = 0 \quad \beta = \frac{4}{3} \quad \alpha = -\frac{1}{3} \]

\[ 2\delta + \frac{\ell}{2} = 0 \quad 4\delta + \frac{\ell}{4} = 1 \quad \delta = \frac{4}{15} \quad \ell = -\frac{4}{15} \]

\[ U = \left( \frac{4}{3} r - \frac{r}{3} \right) \cos \theta + \frac{4}{15} \left( r^2 - \frac{1}{r^2} \right) \sin(2\theta) \]

(2) \[ x u_x - y u_y = x e^x \quad u(x,0) = e^{2x}, \quad x > 0 \]

\[ \begin{align*}
\dot{x} &= x \\
\dot{y} &= -y \\
xy &= c
\end{align*} \]

\[ \frac{du}{dt} = c e^t \exp(c e^t) \]

\[ u = \exp(c e^t) + \tilde{F}(c) \]

\[ u(x,y) = e^x + \tilde{F}(x y) \]

\[ u(x,x) = e^x + \tilde{F}(x^2) = e^{2x} \]

\[ \tilde{F}(x^2) = e^{2x} - e^x, \quad x > 0 \]

\[ \tilde{F}(x) = \frac{e^{2x} - e^x}{x} \]

\[ u(x,y) = e^x + \frac{2 \sqrt{xy}}{\sqrt{xy}} \]

\[ x = a e^t, \quad y = b e^{-t} \]

\[ x = c e^t, \quad y = e^t \]
11. \[ u_{tt} = u_{xx}, \quad -\infty < x < \infty \]
\[ u(x,0) = \begin{cases} 0 & \text{if } t = 0 \\ f(x) & \text{if } t = 0 \end{cases} \]
\[ u_t(x,0) = x e^{-x^2} = g(x) \]

\[ u(x,t) = \frac{1}{2} \left[ f(x+t) + f(x-t) \right] + \frac{1}{2} \int_{x-t}^{x+t} s e^{-s^2} \, ds \]

\[ = \frac{1}{2} \left( -\frac{1}{2} \right) e^{-s^2} \bigg|_{x-t}^{x+t} = \frac{1}{4} \left[ e^{-t^2} - e^{-(x+t)^2} \right] \]

7. ODE \[ t^2 y'' + 3t y' + y = t^3 + \log t \]

General solution \[ y_H = t^r \]
\[ y = y_H + y_p \]
\[ r(r-1)+3r+1=0 \]
\[ r=\begin{cases} -1 & \text{double root} \\ r_1 & \text{and } r_2 \end{cases} \]
\[ y_H = \frac{C_1}{t} + \frac{C_2}{t} \log t \]

Try \[ y_p = A t^3 + B \log t + C \]
\[ t^2 \left[ 6t A - \frac{B}{t^2} \right] + 3t \left[ 3 A t^2 + \frac{B}{t} \right] \]
\[ + A t^3 + B \log t + C = t^3 + \log t \]
\[ (6+9+1) A = 1 \]
\[ B = 1 \]
\[ -8 + 3B + C = 0 \]

\[ A = \frac{1}{16} \]
\[ B = 1 \]
\[ C = -2B = -2 \]

\[ y_p = \frac{t^3}{16} + \log t + 2 \]
ODE
\[ x' = y \]
\[ y' = 4x - 5x^3 + 8x^5 \]

Equilibria: (0,0) (1,0) (-1,0) (2,0) (-2,0)

(0,0): \[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]
E-values: \pm 2 saddle

(\pm 1,0): \[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]
E-values: \pm \sqrt{6} i center

(\pm 2,0): \[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 24 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]
E-values: \pm 2\sqrt{6} = \pm 2\sqrt{6} saddles

\[ \frac{dy}{dx} = \frac{4x - 5x^3 + 8x^5}{y} \]
\[ \frac{y^2}{2} - 2x^2 + \frac{5}{4} x^4 - x^6 = E = \text{energy} \]

(\pm 1,0) are non-linear center

If \( E = -2 + \frac{5}{4} - \frac{1}{6} \) \frac{12 - \frac{45}{6}}{6} = \frac{4}{3}

If \( E = -2 + \frac{5}{4} - \frac{1}{6} \)

\[ U(0) = 0 \]
\[ U(\pm 2) = 12 - \frac{45}{6} = \frac{4}{3} \]
9. \( y'' + xy' + y = 0 \)

\( y = e^{-x^2} \quad y' = -xy \quad y'' = -xy' - y \)

obtain solutions as Taylor series about \( x = \infty \)
try to sum in closed form

\[
Y = \sum_{n=0}^{\infty} q_n x^n \\
\sum_{n=0}^{\infty} (n+2)(n+1)q_{n+2} x^n + \sum_{n=0}^{\infty} nq_n x^n + \sum_{n=0}^{\infty} q_n x^n = 0
\]

\[(n+2)(n+1)q_{n+2} + (n+1)q_n = 0\]

if \( q_1 = 0 \) then \( q_3 = q_5 = q_7 = \ldots = 0 \)

\[q_2 = \left(\frac{-1}{2}\right)q_0 \quad q_4 = -\frac{1}{4}\frac{1}{8}q_2 = \frac{1}{8}q_0\]

\[q_6 = \frac{1}{6}\cdot\frac{1}{48}q_4 = -\frac{q_0}{488} \quad q_8 = \frac{1}{8}\frac{1}{48}q_0 = \frac{q_0}{4! 2^4}\]

\[Y = \sum_{m=0}^{\infty} \frac{(-1)^m q_0}{(2m)!} x^{2m} = q_0 e^{-x^2/2}\]

if \( q_0 = 0 \) then \( q_2 = q_4 = q_6 = \ldots = 0 \)

\[q_{n+2} = -\frac{1}{(n+2)}q_n \quad q_3 = -\frac{1}{3}q_1 \quad q_5 = \frac{1}{5!}q_1\]

\[Y = q_1 \left[ x - \frac{1}{3} x^3 + \frac{1}{5!} x^5 - \frac{1}{7.5!} x^7 + \ldots \right]
\]

\[= q_1 e^{-x^2/2} \int_0^x e^{s^2/2} ds = q_1 e^{-x^2/2} \int_0^x \left( 1 + \frac{s^2}{2} + \frac{s^4}{4!} + \frac{s^6}{8!} + \ldots \right) ds
\]

\[= q_1 e^{-x^2/2} \left[ x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \right] \]