Math 180 **Name (Print):** ____________________
Fall 2014
Final Exam
12/11/2014
Time Limit: 2 Hours

This exam contains 17 pages (including this cover page) and 13 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

The following rules apply:

- You are expected to abide by the University's rules concerning Academic Honesty.
- You may *not* use your books, notes, or any electronic device including cell phones.
- You must show all of your work. An answer, right or wrong, without the proper justification will receive little to no credit.
- You must complete your work in the space provided. No extra paper will be provided.

Circle your instructor.

- Adrovic @ 11am
- Adrovic @ 1pm
- Cabrera
- Dumas
- Kashcheyeva
- Kobotis
- Lowman
- Shin @ 10am
- Shin @ 2pm
- Shulman @ 8am
- Shulman @ 9am

Problem	Points	Score
1	15	
2	10	
3	10	
4	10	
5	10	
6	10	
7	20	
8	10	
9	10	
10	10	
11	10	
12	10	
13	15	
Total:	150	

TA Name: ____________________

1. (15 points) Calculate $f''(x)$ if $f(x) = (3x + e^x)\ln(x)$.

SOLUTION:

$$f'(x) = (3 + e^x)\ln(x) + \frac{3x + e^x}{x}$$
$$f''(x) = e^x \ln(x) + \frac{3 + e^x}{x} + \frac{x(3 + e^x) - (3x + e^x)}{x^2}$$

OR

$$f'(x) = (3 + e^x)\ln x + (3x + e^x)x^{-1}$$
$$f''(x) = e^x \ln(x) + \frac{3 + e^x}{x} + (3 + e^x)x^{-1} - x^{-2}(3x + e^x)$$

GRADING RUBRIC:

If an attempt is made to use the product rule to find $f'(x)$:

2 points – If the student found the derivative of $3x + e^x$

1 points – If the student found the derivative of $\ln x$

2 points – Only if the student found the derivative of both terms and applied the product rule

If an attempt is made to use the product/quotient rule to find $f''(x)$:

2 points – If the student found the derivative of $3 + e^x$

1 point – If the student found the derivative of $\ln x$

2 points – Only if the student found the derivative of both terms and applied the product rule to differentiate $(3 + e^x)\ln x$

2 points – If the student found the derivative of $3x + e^x$

1 point – If the student found the derivative of x (if (s)he used the quotient rule) or the derivative of x^{-1} (if (s)he used the product rule)

2 points – Only if the student found the derivative of both terms and applied the quotient rule to differentiate $\frac{3x+e^x}{x}$ or the product rule to differentiate $(3x + e^x)x^{-1}$

2. (10 points) Calculate the following limit. $\lim_{x\to\infty} \frac{x^2-3x+1}{4^x}$

SOLUTION: The notation "L'H" denotes the use of L'Hôpital's rule.

$$\begin{array}{rcl} \lim_{x\to\infty} \frac{x^2-3x+1}{4^x} & = & \frac{\infty}{\infty} \\ & \Downarrow \text{L'H} & \\ \lim_{x\to\infty} \frac{2x-3}{4^x\ln(4)} & = & \frac{\infty}{\infty} \\ & \Downarrow \text{L'H} & \\ \lim_{x\to\infty} \frac{2}{4^x[\ln(4)]^2} & = & \frac{2}{\infty} = 0. \end{array}$$

GRADING RUBRIC:

10 points – If the student says "An exponential with base larger than 1 grows faster than any polynomial, so the limit is 0."

OR

1 point – If the student states that the limit is of the form $\frac{\infty}{\infty}$

4 points – If the student states that the limit is of the form $\frac{\infty}{\infty}$, denotes some use of L'Hôpital's rule, and finds the derivative of the numerator and denominator (1 point for numerator and 2 points for denominator)

5 points – After the first application of L'Hôpital's rule, if the student states that the new limit is of the form $\frac{\infty}{\infty}$

8 points – After the first application of L'Hôpital's rule, if the student states that the new limit is of the form $\frac{\infty}{\infty}$, denotes some use of L'Hôpital's rule again, and finds the derivative of the numerator and denominator (1 point for numerator and 2 points for denominator)

10 points – After the second application of L'Hôpital's rule, if the student comes to the correct conclusion

3. (10 points) Consider the integral $\int_1^4 \frac{1}{x}\,dx$.

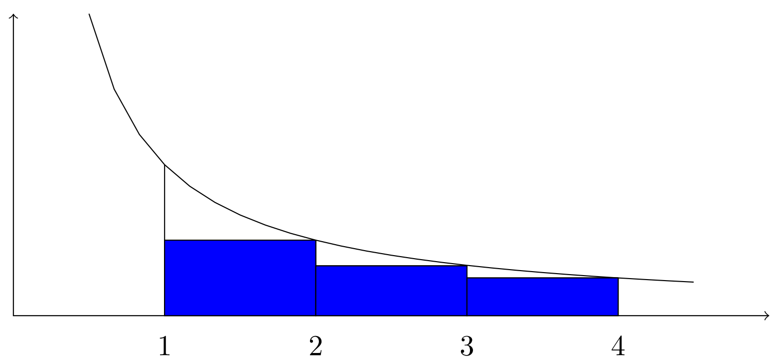
   (a) (6 points) Calculate R_3, the right Riemann sum with 3 subintervals. Write your answer as a single fraction.

   (b) (4 points) Is R_3 an overestimate or underestimate to the value of the definite integral? Justify your answer.

SOLUTION: (a) First, $\Delta x = \frac{4-1}{3} = 1$. Then then 3 subintervals are $[1, 2]$, $[2, 3]$, and $[3, 4]$ so the right endpoints are $x_0^* = 2$, $x_1^* = 3$ and $x_2^* = 4$. Then

$$R_3 = 1 \cdot \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{13}{12}.$$

(b) Since $y = \frac{1}{x}$ is decreasing on $[1, 4]$, the right Riemann sum will be an underestimate. That can also be seen from a picture.



GRADING RUBRIC: (a)

1 point – If the student found Δx, but did not do anything else correct

4 points – If the student set up the right Riemann sum

6 points – If the student set up the right Riemann sum AND computed it as a single fraction

(b)

4 points – If the answer is correct AND proper justification is provided; proper justification would be stating the function is decreasing or a picture of the Riemann sum

4. (10 points) Find all points (x, y) on the curve $x^2+xy+y^2 = 3$ where the tangent line is vertical.

SOLUTION: Using implicit differentiation

$$\begin{aligned} 2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx}(x + 2y) &= -2x - y \\ \frac{dy}{dx} &= \frac{-2x - y}{x + 2y} \end{aligned}$$

The tangent line is then vertical if $\frac{dy}{dx}$ is undefined, or when $x + 2y = 0$ so $x = -2y$. We plug this into the original equation and get

$$\begin{aligned} (-2y)^2 + (-2y)y + y^2 &= 3 \\ 3y^2 &= 3 \\ y &= \pm 1. \end{aligned}$$

Therefore the two points where the tangent line is vertical are $(-2, 1)$ and $(2, -1)$.

GRADING RUBRIC:

1 point – If the student found the derivative of xy

1 point – If the student found the derivative of y^2

1 point – If the student found all derivatives correctly and has $2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

2 points – If the student found that $\frac{dy}{dx} = -2x - yx + 2y$

3 points – If the student set $x + 2y = 0$ and plugged $x = -2y$ into the original equation

2 points – If the student found both points where the vertical tangent line occurs

5. (10 points) Using the definition of the derivative as the limit of a difference quotient, find $\dfrac{dy}{dx}$ if $y = \sqrt{3x}$.

SOLUTION:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{\sqrt{3(x+h)} - \sqrt{3x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3(x+h)} - \sqrt{3x}}{h} \cdot \frac{\sqrt{3(x+h)} + \sqrt{3x}}{\sqrt{3(x+h)} + \sqrt{3x}} \\
&= \lim_{h\to 0} \frac{3(x+h) - 3x}{h\left(\sqrt{3(x+h)} + \sqrt{3x}\right)} \\
&= \lim_{h\to 0} \frac{3}{\left(\sqrt{3(x+h)} + \sqrt{3x}\right)} = \frac{3}{2\sqrt{3x}}
\end{aligned}$$

GRADING RUBRIC:

0 points – If the student does not use the definition of the derivative

2 points – If the student sets up the limit definition using $y = \sqrt{3x}$, but does not continue correctly from here

4 points – If the student sets up the limit definition and knows to multiply by the conjugate of the numerator, but does not multiply correctly

7 points – If the student sets up the limit definition and multiplies by the conjugate to cancel the h's but does not compute the limit correctly

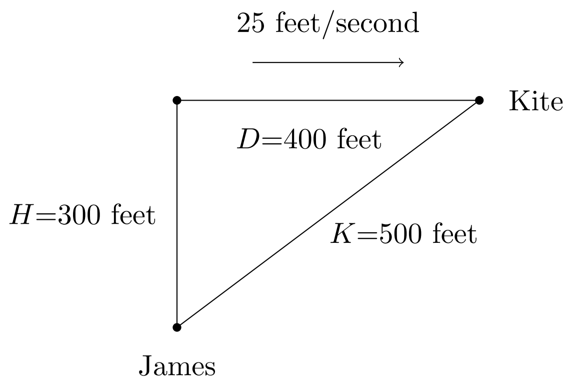
10 points – If the student sets up the limit definition and multiplies by the conjugate to cancel the h's and computes the limit

0 points – If the student did not write a limit anywhere

0 points – If the student used L'Hôpital's rule

6. (10 points) James is flying a kite 300 feet above the ground. A wind gust carries the kite horizontally away from James at a rate of 25 feet per second. How fast must James let the string out when the kite is 500 feet from him?

SOLUTION: Drawing a picture and using the Pythagorean Theorem, we find that D, the horizontal distance between James and the kite, is 400 feet.



Then

$$\begin{aligned} H^2 + D^2 &= K^2 \\ 2H\frac{dH}{dt} + 2D\frac{dD}{dt} &= 2K\frac{dK}{dt} \end{aligned}$$

Since the vertical distance between James and the kite is not changing, we know that $\dfrac{dH}{dt} = 0$. So

$$\begin{aligned} 2(400)(25) &= 2(500)\frac{dK}{dt} \\ \frac{dK}{dt} &= 20. \end{aligned}$$

So James should let the string out at 20 feet/second.

GRADING RUBRIC:

If the student had the correct picture, the following point system was used:

3 points – If the student drew a picture and labelled <u>all</u> the given values in the problem

4 points – If the student drew the picture, labelled it, and used the Pythagorean theorem

7 points – If the student drew the picture, labelled it, used the Pythagorean theorem, and differentiated

9 points – If the student drew the picture, labelled it, used the Pythagorean theorem, differentiated, and plugged in the values

10 points – If the student drew the picture, labelled it, used the Pythagorean theorem, differentiated, plugged in the values, and got the correct answer

If the student had the wrong picture, the following point system was used:

4 points – If the student used the Pythagorean theorem and differentiated correctly

6 points – If the student used the Pythagorean theorem, differentiated correctly, and plugged in the values

Note:

-1 point – If the student made a minor algebra error

-2 points – If the student did not have $\dfrac{dH}{dt} = 0$

7. (20 points) Let $y = \dfrac{10 \ln x}{x^2}$.

   (a) (2 points) State the domain of y.

   (b) (4 points) Find and classify all critical points of y.

   (c) (4 points) Find the intervals where y is increasing and the intervals where y is decreasing.

   (d) (4 points) Find all horizontal asymptotes of y.

   (e) (4 points) Find all vertical asymptotes of y.

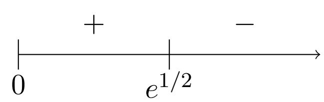
   (f) (2 points) Using your information from parts (a)-(e), sketch a graph of y.

SOLUTION: (a) The domain is $(0, \infty)$.

(b) $y' = \dfrac{x^2 \cdot \frac{10}{x} - 20x \ln x}{x^4}$. The only critical point is when $y' = 0$ since $x > 0$ so

$$\begin{aligned} 10x - 20x \ln x &= 0 \\ \ln x &= \frac{1}{2} \\ x &= e^{1/2} \end{aligned}$$

is the only critical point. Creating a first derivative sign diagram



we see that $e^{1/2}$ is a local maximum.

(c) y is decreasing on $(e^{1/2}, \infty)$ and y is increasing on $(0, e^{1/2})$.

(d) The only possible horizontal asymptote is towards $+\infty$. Then

$$\lim_{x \to +\infty} \frac{10 \ln x}{x^2} = \frac{\infty}{\infty} \overset{\text{L'H}}{\Rightarrow} \lim_{x \to +\infty} \frac{10}{2x^2} = 0$$

where we used L'Hôpital's rule. Therefore $y = 0$ is a horizontal asymptote.

(e) The only possible vertical asymptote is at $x = 0$. Then

$$\lim_{x \to 0^+} \frac{10 \ln x}{x^2} = \frac{-\infty}{0}$$

and since the denominator is always positive, it must be that

$$\lim_{x \to 0^+} \frac{10 \ln x}{x^2} = -\infty$$

So $x = 0$ is a vertical asymptote.

(f) Graph to come.

GRADING RUBRIC: (a)

2 points – If the student has the correct domain

(b)

2 points – If the student found y'

3 points – If the student found y' and the single critical point

4 points – If the student found y', the single critical point, and correctly classified it using either a sign diagram of the second derivative test

(c)

2 points – If the increasing interval is correct

2 points – If the decreasing interval is correct

(d)

1 point – If the student shows the limit as $x \to +\infty$ is of the form $\frac{\infty}{\infty}$

3 points – If the student shows the limit as $x \to +\infty$ is of the form $\frac{\infty}{\infty}$, uses L'Hôpital's rule, and finds the derivative of the numerator and denominator

4 points – If the student states that $y = 0$ is a horizontal asymptote

(e)

1 point – If the student sets up the limit as $x \to 0^+$, but evaluates it incorrectly

3 points – If the student sets up the limit as $x \to^+$ and evaluates it correctly

4 points – If the student sets up the limit as $x \to^+$, evaluates it correctly, and states that $x = 0$ is a horizontal asymptote

(f)

1 point – If the graph matches some of the information from parts (a)-(e)

2 points – If the graph matches all of the information from parts (a)-(e)

8. (10 points) Calculate the following definite integral. $\displaystyle\int_1^4 \left(\frac{1}{\sqrt{x}} + 6x^2 + 5\right)\, dx$

SOLUTION: An antiderivative of $x^{-1/2}+6x^2+5$ is $F(x) = 2x^{1/2}+2x^3+5x$. By the Fundamental Theorem of Calculus,

$$\int_1^4 \left(\frac{1}{\sqrt{x}} + 6x^2 + 5\right)\, dx = \left(2x^{1/2} + 2x^3 + 5x\right)\Big|_1^4 = 152 - 9 = 143.$$

GRADING RUBRIC:

3 points – If the student integrates $\frac{1}{\sqrt{x}}$

3 points – If the student integrates $6x^2$

2 points – If the student integrates 5

2 points – Only if all three terms were integrated correctly AND the subtraction of the upper and lower limits are correct

Notes:

-1 point – If the student wrote $+C$ and did not cancel it

-1 point – If the student wrote the integral symbol after integrating

9. (10 points) Calculate the following indefinite integral. $\displaystyle\int \frac{20x}{(4x^2+2)^3}\,dx$

SOLUTION: Let $u = 4x^2+2$, so $du = 8x\,dx$, or $\dfrac{1}{8}\,du = x\,dx$. Then

$$\int \frac{20x}{(4x^2+2)^3}\,dx = 20\int \frac{\frac{1}{8}\,du}{u^3} = \frac{5}{2}\cdot\frac{1}{-2}u^{-2} + C = -\frac{5}{4}(4x^2+2)^{-2} + C.$$

GRADING RUBRIC: 2 points – If the student chooses some u and computes du correctly for his/her choice of u, but it is the wrong u-substitution and the rest of the problem is wrong

4 points – If the student chooses $u = 4x^2+2$ and finds $du = 8x\,dx$

6 points – If the student chooses $u = 4x^2+2$, $du = 8x\,dx$ and transforms the integral to $\frac{5}{2}\int\frac{du}{u^3}$

8 points – If the student chooses $u = 4x^2+2$, $du = 8x\,dx$, transforms the integral to $\frac{5}{2}\int\frac{du}{u^3}$, and finds $\frac{5}{2}\int\frac{du}{u^3} = -\frac{5}{4}u^{-2}+C$

10 points – If the student chooses $u = 4x^2+2$, $du = 8x\,dx$, transforms the integral to $\frac{5}{2}\int\frac{du}{u^3}$, finds $\frac{5}{2}\int\frac{du}{u^3} = -\frac{5}{4}u^{-2}+C$, and plugs back in what u is

Notes: Deduct 1 point if the student writes $du = 8x$ and not $du = 8x\,dx$. Deduct 1 point if the student forgets "$+C$".

-1 point – If the sign on the final answer is wrong

10. (10 points) In the following problem a student used u-substitution.

$$\int_e^{e^3} \frac{1}{x\ln(x^2)}\,dx = \int_A^B \frac{1}{2u}\,du$$

(a) (4 points) Which expression in the integral on the left did u replace?

(b) (3 points) Which expression in the integral on the left did du replace?

(c) (3 points) What are the new limits of integration A and B on the integral on the right?

Solution: (a) Here is one solution. The student let $u = \ln(x^2)$.

(b) Then $du = \dfrac{1}{x^2} \cdot 2x$ or $du = \dfrac{2}{x}\,dx$.

(c) The new limits of integration are $A = 2$ to $B = 6$.

Grading Rubric:

If the student chooses $u = \ln(x^2)$, then use the following point system:

(a)

4 points – If the student writes $u = \ln(x^2)$

0 points – Otherwise

(b)

3 points – If the student writes $du = \frac{2}{x}\,dx$. [Note: The student does not need to write $\frac{1}{2}\,du = \frac{1}{x}\,dx$.]

0 points – Otherwise

(c)

0 points – If both A and B are incorrect

1 point – If either A or B is correct (but not both)

3 points – If both A and B are correct

Note: If the student transposes A and B, give them 2 points

If the student rewrites $\ln(x^2)$ as $2\ln x$ (AND WRITES THIS ON HIS/HER PAPER) and chooses $u = \ln(x)$, then use the following point system:

(a)

4 points – If the student writes $u = \ln(x)$

0 points – Otherwise

(b)

3 points – If the student writes $du = \frac{1}{x}\,dx$.

0 points – Otherwise

(c)

0 points – If both A and B are incorrect

1 point – If either A or B is correct (but not both)

3 points – If both A and B are correct

Note: If the student transposes A and B, give them 2 points

11. (10 points) Calculate the following indefinite integral. $\displaystyle\int \frac{x}{x-5}\,dx$

Solution: Let $u = x - 5$, so $du = dx$. Then $x = u + 5$ so

$$\int \frac{u+5}{u}\,du = \int \left(1 + \frac{5}{u}\right)\,du = u + 5\ln|u| + C = x - 5 + 5\ln|x-5| + C$$

Grading Rubric:

If the student uses a substitution then use the following point system:

2 points – If the student chooses some u and computes du correctly for his/her choice of u, but it is the wrong u-substitution and the rest of the problem is wrong

2 points – If the student chooses $u = x - 5$ but does no find du

4 points – If the student chooses $u = x - 5$ and finds $du = dx$

6 points – If the student chooses $u = x - 5$, $du = dx$ and transforms the integral to $\int \frac{u+5}{u}\,du$

9 points – If the student chooses $u = x - 5$, $du = dx$, transforms the integral to $\int \frac{u+5}{u}\,du$, and finds $\int \frac{u+5}{u}\,du = u + 5\ln|u| + C$

10 points – If the student chooses $u = x - 5$, $du = dx$, transforms the integral to $\int \frac{u+5}{u}\,du$, finds $\int \frac{u+5}{u}\,du = u + 5\ln|u| + C$, and plugs back in what u is

-1 point – If the student does not change the dx to du

Notes: Deduct 1 point if the student writes $du = 1$ and not $du = 1 \cdot dx$. Deduct 1 point if the student forgets "$+C$". Deduct 1 point if the student forgets the absolute value symbols inside the natural log function.

If the student added and subtracted 5 in the numerator, then use the following point system:

2 points – If the student added and subtracted 5 in the numerator, but then did not proceed correctly

4 points – If the student added and subtracted 5 in the numerator and split the fraction to $\frac{x-5}{x-5} + \frac{5}{x-5}$

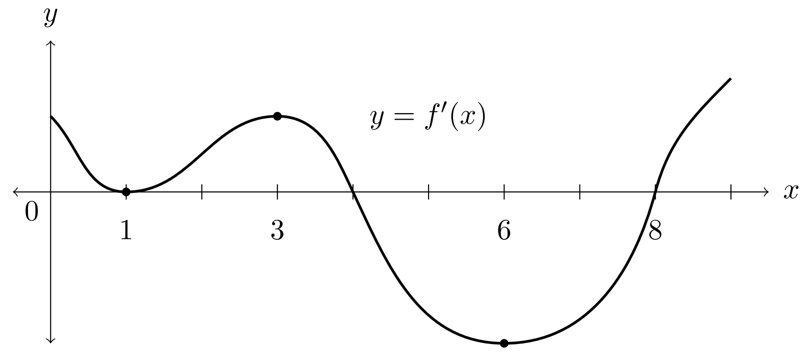
6 points – If the student added and subtracted 5 in the numerator, split the fraction to $\frac{x-5}{x-5} + \frac{5}{x-5}$, and only integrated $\frac{x-5}{x-5}$ correctly, but not $\frac{5}{x-5}$

7 points – If the student added and subtracted 5 in the numerator, split the fraction to $\frac{x-5}{x-5} + \frac{5}{x-5}$, and only integrated $\frac{5}{x-5}$ correctly, but not $\frac{x-5}{x-5}$

9 points – If the student added and subtracted 5 in the numerator, split the fraction to $\frac{x-5}{x-5} + \frac{5}{x-5}$, and integrated $\frac{x-5}{x-5}$ and $\frac{5}{x-5}$ correctly

10 points – If the student added and subtracted 5 in the numerator, split the fraction to $\frac{x-5}{x-5} + \frac{5}{x-5}$, integrated $\frac{x-5}{x-5}$ and $\frac{5}{x-5}$ correctly, and added the terms together with "$+C$"

12. (10 points) Below is the graph of THE DERIVATIVE $f'(x)$; it is NOT $f(x)$.



(a) (5 points) List all critical points of $f(x)$ and classify each as a local minimum, local maximum, or neither.

(b) (5 points) State the intervals where f is concave up and the intervals where f is concave down.

SOLUTION: (a) The critical points and their classifications are

$x = 1$	neither
$x = 4$	local maximum
$x = 8$	local minimum

(b) The function $f(x)$ is concave up on $(1, 3)$ and $(6, 9)$. The function $f(x)$ is concave down on $(0, 1)$ and $(3, 6)$.

GRADING RUBRIC: (a)

0 points – If the student has 0 or 1 critical points listed

1 point – If the student has 2 critical points listed

2 points – If the student has all 3 critical points listed

1 point (per critical point) – If the student classified the critical point

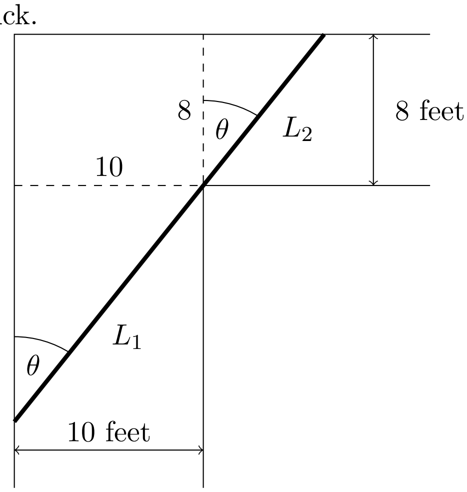
(b)

2 points – If either the concave up or concave down intervals are correct, but not both

5 points – If both the concave up and concave down intervals are correct

13. (15 points) An electrician is carrying a piece of pipe down a hallway that is 10 feet wide. The electrician needs to navigate a right-angled turn to proceed down a hallway that measures 8 feet wide. In this problem you will consider whether the pipe can be carried around the corner or not.

   (a) (5 points) If the pipe gets stuck at some point while navigating the corner, then at the moment it becomes stuck it will be touching two outer walls and the inner corner, as shown in the figure below. (This is a view from above. We assume the pipe is always carried parallel to the floor.) Calculate the length of the pipe $L_1 + L_2$ as a function of the angle θ at which it becomes stuck.



   (b) (10 points) Calculate the minimum length of a pipe that will get stuck while turning the corner.

Solution: (a) From the picture $\sin\theta = \dfrac{10}{L_1}$ so $L_1 = 10\csc\theta$ and $\cos\theta = \dfrac{8}{L_2}$ so $L_2 = 8\sec\theta$. Therefore

$$L_1 + L_2 = 10\csc\theta + 8\sec\theta.$$

(b) We seek the minimum of $L(\theta) = 10\csc\theta + 8\sec\theta$ subject to $0 < \theta < \pi$. Then

$$\begin{aligned}
L'(\theta) = -10\csc\theta\cot\theta + 8\sec\theta\tan\theta &= 0 \\
-\frac{10\cos\theta}{\sin^2\theta} + \frac{8\sin\theta}{\cos^2\theta} &= 0 \\
\frac{-10\cos^3\theta + 8\sin^3\theta}{\sin^2\theta\cos^2\theta} &= 0 \\
-10\cos^3\theta + 8\sin^3\theta &= 0 \\
8\sin^3\theta &= 10\cos^3\theta \\
\tan^3\theta &= \frac{5}{4} \\
\theta &= \tan^{-1}\left(\sqrt[3]{\frac{5}{4}}\right).
\end{aligned}$$

With this angle θ, we have $\csc\theta = \dfrac{\sqrt{1+(5/4)^{2/3}}}{\sqrt[3]{5/4}}$ and $\sec\theta = \sqrt{1+(5/4)^{2/3}}$ so that the minimum length of pipe that will get stuck is

$$\frac{10\sqrt{1+(5/4)^{2/3}}}{\sqrt[3]{5/4}} + 8\sqrt{1+(5/4)^{2/3}} \approx 25.4 \text{ feet.}$$

Grading Rubric: (a)

2 points – If the student finds L_1 or L_2 in terms of θ, but not both

5 points – If the student finds both L_1 and L_2 in terms of θ

(b)

4 points – If the student calculates the derivative $L_1 + L_2$ [Note: If the student does not have $L_1 + L_2$ correct from part (a), then (s)he can earn these 4 points on part (b) but nothing else.]

4 points – If the student finds the single critical point θ

2 points – If the student finds the minimum length