

DO NOT WRITE ABOVE THIS LINE!!

## MATH 180 Final Exam December 8, 2016

Directions. Fill in each of the lines below. Circle your instructor's name and write your TA's name. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR INSTRUCTOR. Good luck!

Print Name: ANSWER KEY

University Email: \_\_\_\_\_

UIN: \_\_\_\_\_

Circle your instructor's name:      Boester      Dumas      Embers      Riedl      Shulman

TA's Name: \_\_\_\_\_

- VERY IMPORTANT!!! CHECK THAT THE NUMBER AT THE TOP OF EACH PAGE OF YOUR EXAM IS THE SAME. IT IS THE NUMBER PRECEDED BY A POUND (#) SIGN. IF THEY ARE NOT ALL THE SAME, NOTIFY YOUR INSTRUCTOR OR TA RIGHT AWAY.
- You must show your work on all problems.
- All of your work must fit within the boxes on each page for each question. Nothing outside of the box will be graded! If you write outside of the box, there is a good chance that your exam will not be read and therefore not graded.
- A solution for one problem may not go on another page.
- Make clear to the grader what your final answer is.
- Have your student ID ready to be checked when submitting your exam.

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1. (19 points) Find the following derivatives. YOU DO NOT NEED TO SIMPLIFY YOUR ANSWERS.

(a)  $\frac{d}{dx} \left( \frac{\cos x}{\ln x} \right)$       quotient rule       $u = \cos x$        $v = \ln x$   
 $u' = -\sin x$        $v' = \frac{1}{x}$   
$$= \frac{(-\sin x)(\ln x) - (\cos x)(\frac{1}{x})}{(\ln x)^2}$$

(b)  $\frac{d}{dx} (\arctan(x^2) \cdot e^x)$       product + chain rule       $u = \arctan(x^2)$        $v = e^x$   
 $u' = \frac{2x}{1+(x^2)^2}$        $v' = e^x$   
$$= \frac{2x}{1+(x^2)^2} \cdot e^x + \arctan(x^2) \cdot e^x$$

(c)  $\frac{d}{dx} \left( \int_0^{x^3} t \cdot 3^t dt \right)$       Fundamental Theorem of Calculus  
+ chain rule

$$A(x) = \int_0^x t \cdot 3^t dt \Rightarrow A'(x) = x \cdot 3^x \quad \text{FTC}$$

$$\begin{aligned} (A(x^3))' &= A'(x^3) \cdot 3x^2 \\ &= x^3 \cdot 3^{x^3} \cdot 3x^2 \end{aligned}$$

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2. (24 points) Find the integrals below. Make sure to justify your work.

$$(a) \int 4e^{3x} dx = \frac{4}{3} e^{3x} + C$$

$$(b) \int \left( \frac{1}{x^3} - \frac{1}{\sqrt{1-x^2}} \right) dx = \frac{-x^{-2}}{2} - \arcsin x + C$$

$\frac{1}{x^3} = x^{-3}$                        $\uparrow$   
power rule

$$(c) \int_1^e \left( \frac{1}{x} - 5 \right) dx = \left( \ln|x| - 5x \right) \Big|_1^e$$
$$= (\ln e - 5e) - (\ln 1 - 5)$$
$$= 1 - 5e + 5$$
$$= 6 - 5e$$

$$(d) \int_{-3}^3 x^5 \cos x dx = 0$$

Note that  $f(x) = x^5 \cdot \cos x$  is an odd function  
( $x^5$  odd &  $\cos x$  even)

For odd functions  $\int_{-a}^a f(x) dx = 0$

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3. (20 points) Calculate each limit (by any method). Be sure to justify any method you use. If the limit does not exist, then you are required to determine whether the limit is  $+\infty$  or  $-\infty$ .

(a)  $\lim_{x \rightarrow +\infty} \frac{\cos x}{\sqrt{x^2 - 1}} = \boxed{0}$

$$-1 \leq \cos x \leq +1$$

$$\Rightarrow \frac{-1}{\sqrt{x^2 - 1}} \leq \frac{\cos x}{\sqrt{x^2 - 1}} \leq \frac{1}{\sqrt{x^2 - 1}}$$

Squeeze Theorem  $\lim(\ ) \leq \lim(\ ) \leq \lim(\ )$

$$\Rightarrow 0 \leq \lim_{x \rightarrow +\infty} \frac{\cos x}{\sqrt{x^2 - 1}} \leq 0$$

$$\Rightarrow \lim_{x \rightarrow +\infty} \frac{\cos x}{\sqrt{x^2 - 1}} = \boxed{0}$$

$$\lim_{x \rightarrow \infty} \pm \frac{1}{\sqrt{x^2 - 1}} = 0$$

$$(\text{of type } \pm \frac{1}{\infty})$$

(b)  $\lim_{x \rightarrow -\infty} x^3 e^{-4x}$

$$= \boxed{-\infty}$$

$$\text{since } \lim_{x \rightarrow -\infty} x^3 = -\infty \quad \& \quad \lim_{x \rightarrow -\infty} e^{-4x} = +\infty$$

(c)  $\lim_{x \rightarrow 1} \frac{3x^3 - 12x^2 + 15x - 6}{2x^3 - 6x + 4}$  of type  $\frac{0}{0}$

$$= \lim_{x \rightarrow 1} \frac{9x^2 - 24x + 15}{6x^2 - 6} \text{ of type } \frac{0}{0}$$

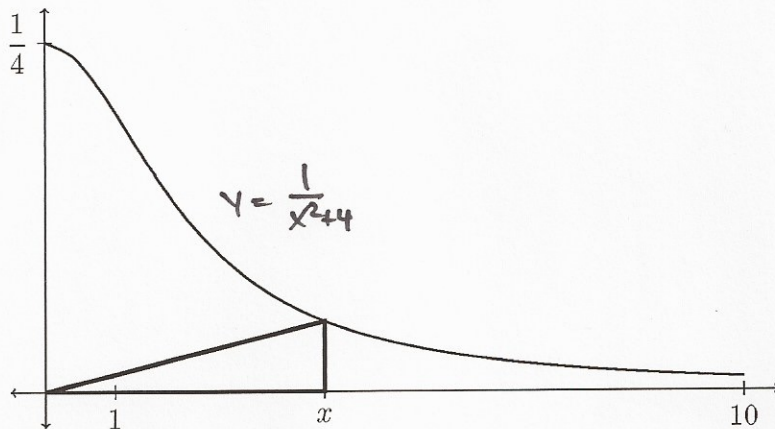
L'Hopital

$$= \lim_{x \rightarrow 1} \frac{18x - 24}{12x} = \frac{-6}{12} = \boxed{-\frac{1}{2}}$$

L'Hopital

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4. (15 points) A right triangle has one vertex at the origin, a second vertex on the positive  $x$ -axis at  $(x, 0)$ , and a third vertex on the curve  $y = \frac{1}{x^2 + 4}$  (see the figure below). If the point  $(x, 0)$  satisfies  $1 \leq x \leq 10$ , what is the maximum and minimum area for such a triangle? Write your maximum and minimum areas in the space provided.



Find max & min area for  $1 \leq x \leq 10$ .

$$f(x) = \text{Area} = \frac{1}{2} \cdot x \cdot \frac{1}{x^2 + 4}$$

$$\text{Domain} = [1, 10]$$

$$f'(x) = \frac{1}{2} \cdot \left( \frac{1 \cdot (x^2 + 4) - x \cdot 2x}{(x^2 + 4)^2} \right) = 0$$

Compare values

$$x^2 + 4 - 2x^2 = 0$$

$$4 - x^2 = 0$$

$$x^2 = 4$$

$$x = \pm 2$$

$$x = +2$$

|    | Area                         |
|----|------------------------------|
| 1  | $\frac{1}{10}$               |
| 2  | $\frac{1}{8} = \text{max}$   |
| 10 | $\frac{5}{104} = \text{min}$ |

Minimum Area:  $\frac{5}{104}$  Maximum Area:  $\frac{1}{8}$

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5. (10 points) Consider a function  $f(x)$  whose first and second derivatives are  $f'(x) = \frac{x^2 - 9}{x^2 + 9}$  and  $f''(x) = \frac{36x}{(x^2 + 9)^2}$ . Make sure to justify your answers with a sign diagram or some other work.

- (a) In the spaces provided, state the intervals where  $f(x)$  is decreasing and those where  $f(x)$  is increasing.

$f' = 0 \quad x = \pm 3$   
 $f' = \phi \quad \text{N/A}$

|              |    |   |   |
|--------------|----|---|---|
| $x^2 - 9$    | +  | - | + |
| $x^2 + 9$    | +  | + | + |
| sign of $f'$ | +  | - | + |
|              | ↗  | ↘ | ↗ |
|              | -3 | 3 |   |

Decreasing:  $(-3, 3)$  Increasing:  $(-\infty, -3) \cup (3, \infty)$

- (b) Find the critical points of  $f(x)$  and classify each as a local minimum, local maximum, or neither.

$f' = 0 \quad x = \pm 3$   
 $f' = \phi \quad \text{N/A}$

@  $x = -3$  local max  
 @  $x = +3$  local min

- (c) In the spaces provided, state the intervals where  $f(x)$  is concave down and those where  $f(x)$  is concave up.

$f'' = 0 \quad x = 0$   
 $f'' = \phi \quad \text{N/A}$

|               |   |   |
|---------------|---|---|
| $36x$         | - | + |
| $(x^2 + 9)^2$ | + | + |
| sign of $f''$ | - | + |
|               | ↘ | ↗ |
|               | 0 |   |

Concave Down:  $(-\infty, 0)$  Concave Up:  $(0, \infty)$

- (d) List any inflection points.

$f$  has an inflection point at  $x = 0$ .

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6. (8 points) Consider the function  $f(x) = x^2$  on the closed interval  $[1, 4]$ . Using the theorems below, complete each of the statements in (a)-(d). Each theorem is used at most once.

- Mean Value Theorem for Integrals
- Mean Value Theorem
- Extreme Value Theorem
- Rolle's Theorem
- Intermediate Value Theorem

(a) The Intermediate Value Theorem says that there is a  $c$  in  $(1, 4)$  with  $f(c) = k$  where  $k$  is any number between 1 and 16.

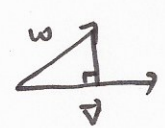
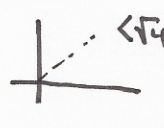
(b) The Mean Value Theorem says that there is a  $c$  in  $(1, 4)$  with  $f'(c) = \frac{16-1}{4-1}$ .

(c) The Mean Value Theorem for Integrals says that there is a  $c$  in  $(1, 4)$  with  $f(c) = \frac{\frac{4^3}{3} - \frac{1^3}{3}}{4-1}$ .

(d) The Extreme Value Theorem says that there is an absolute maximum and absolute minimum for  $f(x)$  on  $[1, 4]$ .

7. (10 points) Consider the vectors  $\mathbf{v} = \langle 2, 1 \rangle$  and  $\mathbf{w} = \langle -1, 1 \rangle$ . In the grid below, answer "YES" or "NO". In the first column answer if the given vector is PARALLEL to  $\mathbf{v}$ , and in the second column answer if the given vector is ORTHOGONAL to  $\mathbf{w}$ .

| Given Vector                                                                            | PARALLEL to $\mathbf{v}$ ? | ORTHOGONAL to $\mathbf{w}$ ? |
|-----------------------------------------------------------------------------------------|----------------------------|------------------------------|
| $3\mathbf{v}$                                                                           | YES                        | NO                           |
| the projection of $\mathbf{w}$ on $\mathbf{v}$ , $\text{proj}_{\mathbf{v}}(\mathbf{w})$ | YES                        | NO                           |
| the vector 45 degrees north of east with magnitude 4                                    | NO                         | YES                          |
| $4\mathbf{v} + 2\mathbf{w}$                                                             | NO                         | YES                          |
| $\langle -3, -\frac{3}{2} \rangle$                                                      | YES                        | NO                           |

$$\langle \sqrt{4}, \sqrt{4} \rangle \cdot \vec{w} = 0$$

$$4\vec{v} + 2\vec{w} = \langle 8, 4 \rangle + \langle -2, 2 \rangle$$

$$= \langle 6, 6 \rangle \quad (\cdot \langle -1, 1 \rangle = 0)$$

$$\Rightarrow \perp \langle -1, 1 \rangle$$

$$\langle -3, -\frac{3}{2} \rangle = \langle 2, 1 \rangle \cdot \langle -\frac{3}{2} \rangle$$

$$\langle -3, -\frac{3}{2} \rangle \cdot \langle -1, 1 \rangle \neq 0$$

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8. (12 points) Consider the function  $f(x) = \sqrt{x}$ .

(a) Find the linear function that best approximates  $f(x)$  at  $a = 100$ .

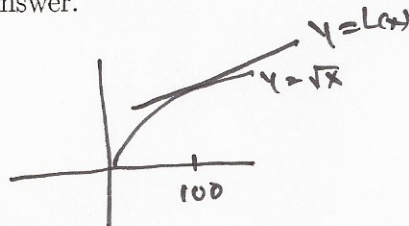
$$f'(x) = \frac{1}{2\sqrt{x}} \quad f(100) = \sqrt{100} = 10 \quad \& \quad f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$$

$$\Rightarrow L(x) = 10 + \frac{1}{20}(x - 100)$$

(b) Using your answer to part (a), estimate  $\sqrt{102}$ .

$$\begin{aligned} \sqrt{102} &\approx 10 + \frac{1}{20}(\underbrace{102 - 100}_2) \\ &= 10.1 \end{aligned}$$

(c) Is your estimate in part (b) an overestimate or underestimate to the exact value of  $\sqrt{102}$ ? Justify your answer.

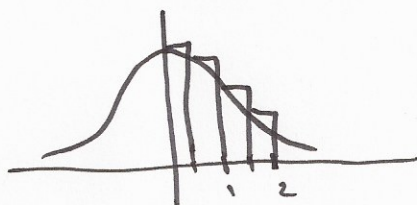


OVER ESTIMATE  
since  $y = L(x)$  is above  
the graph  $y = \sqrt{x}$ .

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9. (12 points) Consider the function  $f(x) = \frac{1}{1+x^2}$

(a) Calculate the left hand Riemann sum for  $f(x)$  on  $[0, 2]$  with 4 rectangles. Write your final answer as a reduced fraction.



| $x$    | 0 | $\frac{1}{2}$ | 1             | $\frac{3}{2}$  | 2             |
|--------|---|---------------|---------------|----------------|---------------|
| $f(x)$ | 1 | $\frac{4}{5}$ | $\frac{1}{2}$ | $\frac{4}{13}$ | $\frac{1}{5}$ |

$$\begin{aligned}
 L_4 &= \left( 1 + \frac{4}{5} + \frac{1}{2} + \frac{4}{13} \right) \cdot \frac{1}{2} \\
 &= \frac{130 + 104 + 65 + 40}{130} \cdot \frac{1}{2} \\
 &= \frac{339}{130} \cdot \frac{1}{2} = \frac{339}{260}
 \end{aligned}$$

(b) Is your Riemann sum in part (a) an overestimate or an underestimate to the actual area? Justify your answer.

It is an overestimate since  $f' = \frac{-2x}{(1+x^2)^2} < 0$  for  $x > 0$ , thus  $f$  is decreasing for  $x > 0$ .

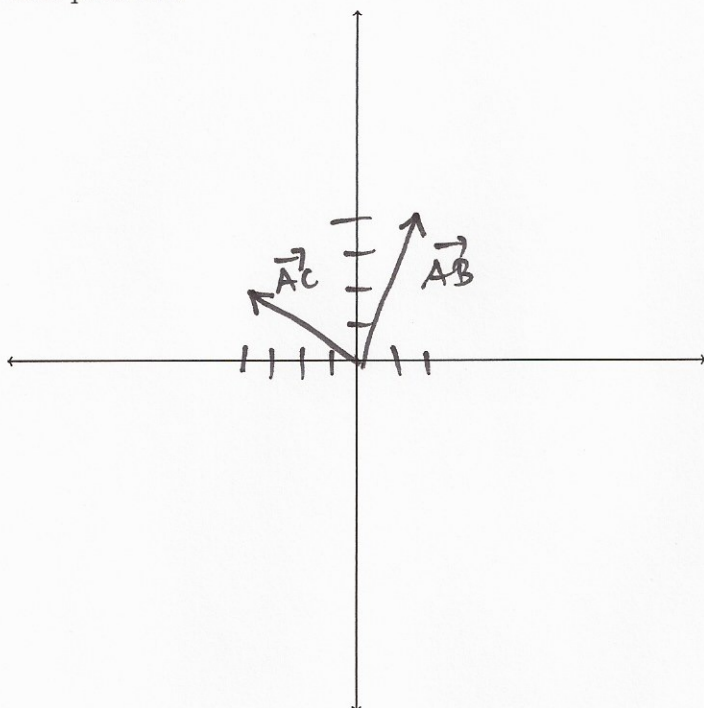
(c) Find the actual area on  $[0, 2]$ . You may leave your answer unsimplified.

$$\begin{aligned}
 \int_0^2 \frac{dx}{1+x^2} &= \arctan x \Big|_0^2 = \arctan 2 - \arctan 0 \\
 &= \arctan 2
 \end{aligned}$$

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10. (10 points) Consider the points  $A(1, 1)$ ,  $B(3, 5)$ , and  $C(-3, 3)$ .

- (a) Write the vectors  $\vec{AB}$  and  $\vec{AC}$  in the form  $\langle a, b \rangle$ . Then sketch and label both of them on the axes provided.



$$\vec{AB} = \langle 2, 4 \rangle$$

$$\vec{AC} = \langle -4, 2 \rangle$$

- (b) Calculate the magnitude of  $\vec{AB}$  and the magnitude of  $\vec{AC}$ .

$$\|\vec{AB}\| = \|\langle 2, 4 \rangle\| = \sqrt{4+16} = \sqrt{20}$$

$$\|\vec{AC}\| = \|\langle -4, 2 \rangle\| = \sqrt{16+4} = \sqrt{20}$$

- (c) Find the angle  $\theta$  between  $\vec{AB}$  and  $\vec{AC}$ . You can leave  $\theta$  unsimplified.

$$\cos \theta = \frac{\vec{AB} \cdot \vec{AC}}{\|\vec{AB}\| \cdot \|\vec{AC}\|} = \frac{2 \cdot (-4) + 4 \cdot 2}{\sqrt{20} \cdot \sqrt{20}} = 0 \Rightarrow \theta = \pi/2 \text{ or } 90^\circ$$

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11. (10 points) Consider the curve defined by

$$x \cos(y) + 4y = 4\pi.$$

Write the equation of the tangent line to the curve at the point  $(0, \pi)$ .

implicit differentiation

product rule :  $\frac{d}{dx} (x \cos y) = 1 \cdot \cos y + x \cdot (-\sin y) \frac{dy}{dx}$

$$\Rightarrow \cos y - x \sin y \frac{dy}{dx} + 4 \frac{dy}{dx} = 0$$

@  $(0, \pi)$  plug in  $x=0$  &  $y=\pi$

$$\underbrace{\cos(\pi)}_{-1} - 0 \cdot \sin \pi \frac{dy}{dx} + 4 \frac{dy}{dx} = 0$$

$$\Rightarrow 4 \frac{dy}{dx} = +1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{4} = \text{slope}$$

$$\Rightarrow \text{tangent line} \quad y - \pi = \frac{1}{4} (x - 0)$$

$$\Rightarrow \boxed{y = \frac{1}{4}x + \pi}$$