MATH 180 Final Exam
December 13, 2018

Directions. Fill in each of the lines below. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR EXAM PROCTOR. This exam contains 12 pages (including this cover page) and 15 problems. After starting the exam, check to see if any pages are missing. Enter all requested information on this page. You are expected to abide by the University's rules concerning Academic Honesty.

TA Name:________________________

The following rules apply:

- You may *not* use your books, notes, calculators, or any electronic device including cell phones. Only pencils/pens allowed.
- You must show all of your work. An answer, right or wrong, without the proper justification will receive little to no credit.
- You *must* complete your work in the space provided. We will be scanning your answers into our grading system, so any work you do that is out of place, too close to the page border, or on the wrong page will *not* be graded!

Circle your instructor.

- Martina Bode
- Mercer (Tabes) Bridges
- Nathan Jones
- Matthew Lee
- John Steenbergen

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1. (18 points) Circle the correct answer. No explanations needed!

(a) (3 points) If f is a continuous function on $[1,5]$ such that $f(1) = 5, f(5) = 2$, then by the Intermediate Value Theorem there is a c in $(1,5)$ with $f(c) = 0$.

True **False**

for ex $f = +5 - \frac{3}{4}(x-1)$
but $f \neq 0$ for $1 \leq x \leq 5$

(b) (3 points) Suppose f is a continuous function on $[1,5]$, and differentiable on $(1,5)$, such that $f(1) = 5, f(5) = 1$, then by the Mean Value Theorem there is a c in $(1,5)$ with $f'(c) = -1$.

True **False**

$f'(c) = \frac{f(5)-f(1)}{5-1} = \frac{1-5}{5-1} = -1$ ✓

(c) (3 points) If $\int_2^9 f(x) \, dx = 3$ then the average value of $f(x)$ on the interval $[2,9]$ is equal to 3/7.

True **False**

$Ave = \frac{1}{9-2} \int_2^9 f(x) dx = \frac{3}{7}$

(d) (3 points) If $2 \leq f(x) \leq 2 + x^2$ for all x, then by the Squeeze Theorem $\lim_{x \to 0} f(x) = 2$.

True **False**

(e) (3 points) A function f is defined by

$$f(x) = \begin{cases} d & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{cases}$$

Which constant d makes f continuous on $(-\infty, \infty)$.

0 **1** **2** **None of these answers**

(f) (3 points) If f is a continuous function with $f(0) = 1$, $f(2) = 2$, and $f(4) = 3$, then for $A(x) = \int_0^{x^2} f(t)dt$ evaluate $A'(2)$ $= f(x^2) \cdot 2x = f(4) \cdot 2 \cdot 2 = 3 \cdot 4 = 12$

2 **4** **8** **None of these answers**

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2. (10 points) Let $f(x) = x^2 - 3$. Using the **limit definition** of the derivative, compute $f'(x)$. No credit will be given for any other approach.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 \cancel{-3} - (x^2 \cancel{-3})}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{-x^2}}{h}$$

$$= \lim_{h \to 0} (2x + h)$$

$$= \boxed{2x}$$

3. (10 points) Find the slope of the tangent line to $xy^2 + \sin y + x^3 = 8$ at the point $(2, 0)$.

implicit diff

product rule for xy^2 : $y^2 + 2xyy'$

$$\Rightarrow \quad y^2 + 2xyy' + (\cos y)y' + 3x^2 = 0$$

@ $(2,0)$

$$\Rightarrow \quad (0 + 0 + y' + 3 \cdot 4) = 0$$

$$\boxed{y' = -12}$$

4. (10 points) Use the following steps to approximate $\tan(0.001)$.

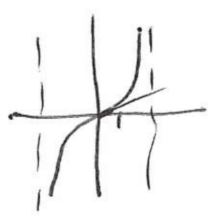
(a) (5 points) Write down the equation of the tangent line to the graph of $f(x) = \tan x$ at $x = 0$.

$$\left.\begin{array}{l} f'(x) = \sec^2 x \\ f'(0) = 1 \\ f(0) = 0 \end{array}\right\} \quad y - 0 = 1\cdot(x-0) \implies \boxed{y = x}$$

(b) (2 points) Use your answer from part (a) to approximate $\tan(0.001)$.

$\tan(0.001) \approx 0.001$ since $\tan x \approx x$ for x close to 0.

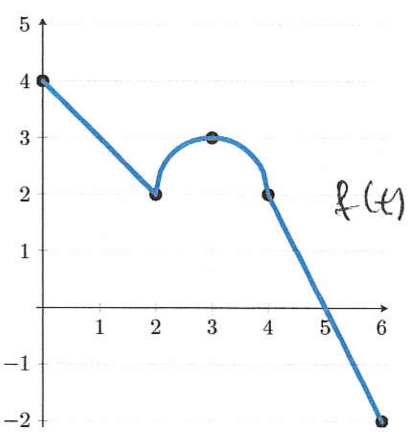
(c) (3 points) Is your answer from part (b) an overestimate or an underestimate? Draw a picture to justify your answer.



underestimate

$\tan x$ concave up for $x > 0$

5. (10 points) The graph of the function $y = f(t)$ is given to the right. Define $g(x) = \int_0^x f(t)\,dt$ on the interval $[0, 6]$.



(a) (5 points) Evaluate $g(0)$, $g(2)$, $g(4)$, $g(6)$, and $g'(3)$.

$g(0) = 0$

$g(2) = 4 + \frac{1}{2}\cdot 2\cdot 2 = 6$

$g(4) = 6 + 4 + \frac{1}{2}\pi$

$g(6) = 10 + \pi/2 + 1 - 1 = 10 + \pi/2$

$g'(3) = f(3) = 3$

(b) (2 points) On which intervals(s) is g increasing?

$(0, 5)$

(c) (3 points) On which interval(s) is g concave up?

on $(2,3)$ only

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6. (22 points) Differentiate the following functions, using logarithmic differentiation if needed. Do not simplify your answers.

(a) (5 points) $f(x) = \arctan(2^x - x)$

chain rule

$$f'(x) = \frac{1}{1+(2^x-x)^2} \cdot \left(\ln 2 \cdot 2^x - 1\right)$$

(b) (7 points) $g(x) = \dfrac{\cos(x^2)}{\ln(x)}$

quotient + chain rule

$$g'(x) = \frac{-2x\sin(x^2)\cdot \ln x - \cos(x^2)\cdot \frac{1}{x}}{(\ln(x))^2}$$

(c) (10 points) $h(x) = x^{\tan(3x)}$

$$\ln h(x) = \tan(3x)\ln x$$

$$\frac{1}{h(x)}\cdot h'(x) = 3\sec^2(3x)\ln x + \frac{\tan(3x)}{x}$$

$$h'(x) = \left(3\sec^2(3x)\ln x + \frac{\tan(3x)}{x}\right)\cdot x^{\tan(3x)}$$

7. (26 points) Compute the following limits, use L'Hôpital's Rule if needed. If a limit is $+\infty$ or $-\infty$ state which one, and if the limit does not exist, explain why.

(a) (6 points) $\lim_{x\to 2} \frac{x^2-4}{x^3-2x^2+x-2}$ of type $\frac{0}{0}$, use L'Hôpital

$= \lim_{x\to 2} \frac{2x}{3x^2-4x+1} = \frac{4}{12-8+1} = \boxed{\frac{4}{5}}$

(b) (6 points) $\lim_{x\to 0} \frac{2+x}{5x}$ of type $\frac{2}{0}$

since $\lim_{x\to 0^-} \frac{2+x}{5x} = -\infty$ and $\lim_{x\to 0^+} \frac{2+x}{5x} = +\infty$ $\Big\}$ $\lim_{x\to 0} \frac{2+x}{5x}$ does not exist

(c) (6 points) $\lim_{x\to\infty} \frac{e^{2x}}{3e^{2x}+2x-7}$ of type $\frac{\infty}{\infty}$, use L'Hôpital

$= \lim_{x\to\infty} \frac{2e^{2x}}{6e^{2x}+2}$ of type $\frac{\infty}{\infty}$, use L'Hôpital again

$= \lim_{x\to\infty} \frac{4e^{2x}}{12e^{2x}} = \frac{4}{12} = \boxed{\frac{1}{3}}$

(d) (8 points) $\lim_{x\to 0}(1+x)^{2/x}$ of type $1^{\pm\infty}$, use ln

$L = \lim_{x\to 0} (1+x)^{2/x}$

$\ln L = \lim_{x\to 0} \frac{2}{x} \cdot \ln(1+x)$

$= \lim_{x\to 0} \frac{2\ln(1+x)}{x}$ "$\frac{0}{0}$" use L'Hôpital

$= \lim_{x\to 0} \frac{2\cdot\frac{1}{1+x}}{1} = 2 \Rightarrow$ limit $L = \boxed{e^2}$

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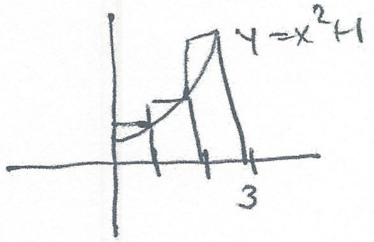
8. (11 points) Let $f(x) = x^2 + 1$.

(a) (5 points) Illustrate and calculate a right-endpoint Riemann sum for $f(x)$ on $[0, 3]$ with $n = 3$ subintervals.



$$R_3 = (2 + 5 + 10) \cdot 1$$
$$= \boxed{17}$$

(b) (6 points) Compute $\int_0^3 f(x)\, dx$.

$$\int_0^3 (x^2+1)\, dx = \frac{x^3}{3} + x \Big|_0^3$$
$$= \left(\frac{3^3}{3} + 3\right) - (0 + 0)$$
$$= 9 + 3 = \boxed{12}$$

9. (5 points) Compute $\int_{-\pi/4}^{\pi/4} \sin^3 x\, dx$. Explain your reasoning!

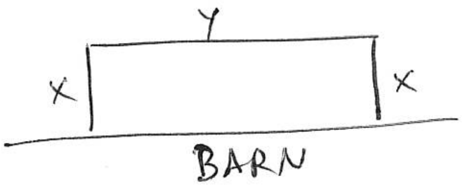
$$= \boxed{0}$$

$\sin x \rightsquigarrow$ odd function

$\sin^3 x \rightsquigarrow$ odd

$\Rightarrow$ the integral is zero b/c of odd symmetry

10. (12 points) A farmer wants to build a rectangular pen along one side of a barn by putting up three sides of fencing. The fence needs to enclose a region measuring 800 square feet. Find the dimensions of such a pen using the smallest amount of fencing possible. Find the domain of your objective function, and justify your answer.



Min $\underbrace{\text{fence}}_{2x+y}$ subject to $\underbrace{\text{area} = 800}_{xy=800}$

$y = \frac{800}{x}$

$\Rightarrow f(x) = 2x + \frac{800}{x}$ Domain = $(0, \infty)$

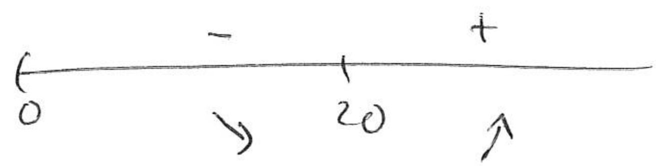
$f'(x) = 2 - \frac{800}{x^2}$

$f' = 0 \Rightarrow 2 = \frac{800}{x^2}$

$\Rightarrow x^2 = 400$

$\Rightarrow \cancel{x = -20}$ or $\boxed{x = 20}$

sign of f'



absolute min @ $x = 20$

$y = \frac{800}{20} = 40$

dimensions are 20 ft × 40 ft

11. (14 points) A bowling ball is dropped from a roof 80 meters above the ground. Assume that the acceleration due to gravity is approximately 10 m/s^2. Show your work using calculus.

(a) (6 points) Find a formula describing the velocity of the bowling ball at time t.

$a = -10$

$v = -10t + c$ since $v_0 = 0 \Rightarrow c = 0$

$\boxed{v = -10t}$

(b) (6 points) Find a formula describing the height of the bowling ball at time t.

$s_0 = 80$

$v = -10t \Rightarrow s = -5t^2 + c$ since $s_0 = 80 \Rightarrow c = 80$

$\boxed{s = -5t^2 + 80}$

(c) (2 points) After how many seconds does the bowling ball hit the ground?

$s = 0 \Rightarrow -5t^2 + 80 = 0$

$5t^2 = 80$

$t^2 = \frac{80}{5} = 16$

$\boxed{t = 4 \text{ seconds}}$

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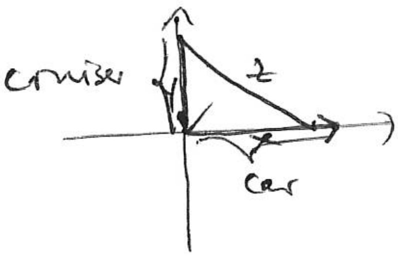
12. (12 points) A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at a rate of 20 mph. If the cruiser is moving at a constant speed of 60 mph, what is the speed of the car? Show all your work.



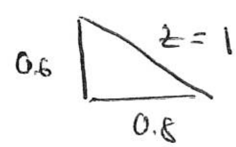
Given: $\frac{dz}{dt} = 20 \text{mph}$

$\frac{dy}{dt} = -60 \text{mph}$

Wand: $\frac{dx}{dt} = ?$

@ $y = 0.6$

$x = 0.8$



$$x^2 + y^2 = z^2$$

$$\cancel{2}x \frac{dx}{dt} + \cancel{2}y \frac{dy}{dt} = \cancel{2}z \frac{dz}{dt}$$

$$(0.8) \frac{dx}{dt} + (0.6)(-60) = (1)(20)$$

$$0.8 \frac{dx}{dt} = 20 + 36$$

$$\frac{dx}{dt} = \frac{56}{0.8} = \frac{560}{8} = \boxed{70 \text{ mph}}$$

13. (14 points) Consider $f(x) = x^3 - 9x^2 + 24x + 5$.

(a) (10 points) Find the x-values at which the absolute minimum and maximum occurs for f on the interval $[1, 3]$.

$f'(x) = 3x^2 - 18x + 24 = 0$

$x^2 - 6x + 8 = 0$

$(x-4)(x-2) = 0$

(i) $f' = 0$ $\quad x = 2 \quad \cancel{x=4}$

(ii) $f' = \varnothing$ $\quad$ N/A

(iii) endpts $x = 1 \quad x = 3$

$\boxed{\begin{array}{ll} @\, x=1 & \text{min value} = 21 \\ @\, x=2 & \text{max value} = 25 \end{array}}$

Compare values

	f
1	$1 - 9 + 24 + 5 = 21$
2	$8 - 36 + 48 + 5 = 25$
3	$27 - 81 + 72 + 5 = 23$

(b) (4 points) The graph of f has a single inflection point. Locate it and justify your answer.

$f''(x) = 6x - 18$

$f'' = 0 @ x = 3$ $\quad$ sign of f'' $\quad$ $-$ | $+$ at 3

f is concave down for $x < 3$, and concave up $x > 3$

$\Rightarrow \boxed{@\, x = 3 \text{ point of inflection}}$

14. (18 points) Let $\vec{u} = \langle 2, 1\rangle$, $\vec{v} = \langle -3, 6\rangle$, and $\vec{w} = \langle 4, 0\rangle$ be vectors in the plane.

(a) (4 points) Compute $3\vec{u} - 2\vec{v}$.

$$3\vec{u} = \langle 6, 3\rangle$$
$$2\vec{v} = \langle -6, 12\rangle$$
$$3\vec{u} - 2\vec{v} = \langle 12, -9\rangle$$

(b) (4 points) Compute $\vec{u} \cdot \vec{v}$.

$$\vec{u} \cdot \vec{v} = 2 \cdot (-3) + 1 \cdot 6 = -6 + 6 = 0$$

(c) (4 points) What does your answer in part (b) say about the relative orientation of the two vectors?

$\vec{u}$ and $\vec{v}$ are perpendicular b/c $\vec{u} \cdot \vec{v} = 0$

(d) (6 points) Find the vector projection of $\vec{v}$ onto $\vec{w}$.

$$\text{proj}_{\vec{w}} \vec{v} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \cdot \vec{w} = \frac{-12}{16} \cdot \langle 4, 0\rangle$$
$$= \langle -3, 0\rangle$$

15. (8 points) If 50 Newtons of force is applied to an object at an angle of $\frac{\pi}{3}$ to the ground over a distance of 20 meters, find the work done on the object.

$$\text{Work} = \vec{F} \cdot \vec{D} = \|\vec{F}\| \cdot \|\vec{D}\| \cdot \cos \pi/3$$
$$= 50\,N \cdot 20\,m \cdot \frac{1}{2}$$
$$= 500 \text{ Joules or Newton-meters}$$