MATH 180 Final Exam

December 9, 2019

Directions. Fill in each of the lines below. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR EXAM PROCTOR. This exam contains 14 pages (including this cover page) and 16 problems. After starting the exam, check to see if any pages are missing. Enter all requested information on this page. You are expected to abide by the University's rules concerning Academic Honesty. Please put your initials on each page.

Answers

Your Name:________________________

Your UIN:________________________

UIC Email:________________________

TA Name:________________________

Circle your instructor.

- Martina Bode
- Shavila Devi
- Vi Diep
- Matthew Lee
- John Steenbergen

The following rules apply:

- You may *not* use your books, notes, calculators, or any electronic device including cell phones. Only pencils/pens allowed.
- You must show all of your work. An answer, right or wrong, without the proper justification will receive little to no credit.
- You *must* complete your work in the space provided. We will be scanning your answers into our grading system, so any work you do that is out of place, too close to the page border, or on the wrong page will *not* be graded!

1. (15 points) MULTIPLE CHOICE

   Circle the correct answer.

   (a) (3 points) If $2 - x^2 \leq f(x) \leq -2x + 3$ for all x, then by the Squeeze Theorem evaluate $\lim_{x \to 1} f(x)$.

   $1 = \lim_{x \to 1}(2 - x^2) \leq \lim f(x) \leq \lim_{x \to 1}(-2x+3) = 1$

   0 [✓ **1**] **2** **None of these answers**

   (b) (3 points) A function f is defined by

   $$f(x) = \begin{cases} 2c & \text{if } x < 0 \\ 4\cos x & \text{if } x \geq 0 \end{cases}$$

   f is continuous on $(-\infty, \infty)$ for $c =$?

   $\lim_{x \to 0^-} f = 2c \qquad \lim_{x \to 0^+} f = 4 \Rightarrow c = 2$

   0 **1** [✓ **2**] **None of these answers**

   (c) (3 points) Evaluate $\frac{d}{dx}\left(\int_0^{x^3} (\sin^3 t)\, dt\right)$.

   $\sin^3(x^3)$ $\cos^3(3x^2)$ $\sin^3(3x^2)$ $3x^2\cos^3(x^3)$ [✓ $3x^2\sin^3(x^3)$]

   (d) (3 points) Given that $f(2) = 4$, and $f'(x) \leq -2$ for all x, can $f(3) = 5$?

   $f(3) \leq 2$ by MVT

   Yes [✓ **No**]

   (e) (3 points) Suppose f is a continuous function. If $\int_2^9 f(x) = 7$ then by the Mean Value Theorem for Integrals there is a c in $(2, 9)$ with $f(c) =$?

   $f(c) = \text{ave value} = \frac{1}{7} \cdot 7 = 1$

   [✓ **1**] **7** **14** **None of these answers**

2. (20 points) Compute the following limits, use L'Hôpital's Rule if needed. If a limit is $+\infty$ or $-\infty$ state which one, and if the limit does not exist, explain why.

(a) (6 points) $\lim_{x\to-3} \frac{\sqrt{x+7}-2}{x+3}$ of type $\frac{0}{0}$, use conjugate

$$= \lim_{x\to-3} \frac{(\sqrt{x+7}-2)(\sqrt{x+7}+2)}{(x+3)(\sqrt{x+7}+2)}$$

$$= \lim_{x\to-3} \frac{x\cancel{+7-4}}{\cancel{(x+3)}(\sqrt{x+7}+2)} = \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$$

or L'Hôpital

$$= \lim_{x\to-3} \frac{\frac{1}{2\sqrt{x+7}}}{1} = \frac{1}{2\sqrt{4}} = \boxed{\frac{1}{4}}$$

(b) (6 points) $\lim_{x\to1^-} \frac{x^3+1}{x^3-1}$ of type $\frac{2}{0^-}$

$x<1 \Rightarrow x^3-1<0$

$$= \boxed{-\infty}$$

(c) (8 points) $\lim_{x\to0^+} (1+2x)^{3/x}$ of type $1^{\pm\infty}$, use natural log

$$L = \lim_{x\to0^+} (1+2x)^{3/x}$$

$$\ln L = \lim_{x\to0^+} \frac{3}{x}\cdot\ln(1+2x)$$ of type $\infty\cdot0$, write as fraction

$$\ln L = \lim_{x\to0^+} \frac{3\ln(1+2x)}{x}$$ of type $\frac{0}{0}$

$$\underset{\text{L'Hôpital}}{=} \lim_{x\to0^+} \frac{3\cdot\frac{2}{1+2x}}{1} = \frac{6}{1} = 6 \Rightarrow \boxed{L=e^6}$$

3. (10 points) Use the **limit definition of derivatives** to find $f'(x)$ for the function $f(x) = 4x^2 - 5$. No credit will be given for using any other method.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(4(x+h)^2 - 5) - (4x^2 - 5)}{h}$$

$$= \lim_{h \to 0} \frac{4(x^2 + 2xh + h^2) \cancel{-5} - 4x^2 \cancel{+5}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4x^2} + 8xh + 4h^2 \cancel{-4x^2}}{h}$$

$$= \lim_{h \to 0} (8x + 4h)$$

$$= 8x$$

4. (20 points) Differentiate the following functions, using logarithmic differentiation if needed. Do not simplify your answers.

(a) (6 points) $f(x) = x^2 \arctan(4x)$

product rule

$u = x^2 \qquad v = \arctan(4x)$

$u' = 2x \times v' = \frac{4}{x^2+1}$

$$f'(x) = 2x \arctan(4x) + x^2 \cdot \frac{4}{x^2+1}$$

(b) (6 points) $g(x) = \frac{e^{2x}}{x^2+1}$

quotient rule

$u = e^{2x} \qquad v = x^2+1$

$u' = 2e^{2x} \times v' = 2x$

$$g'(x) = \frac{(2e^{2x})(x^2+1) - e^{2x} \cdot 2x}{(x^2+1)^2}$$

(c) (8 points) $h(x) = (2 + \sin x)^{3x}$

logarithmic diff

$$\ln h(x) = 3x \ln(2+\sin x)$$

$$\frac{1}{h(x)} \cdot h'(x) = 3 \ln(2+\sin x) + 3x \cdot \frac{\cos x}{2+\sin x}$$

$$h'(x) = \left(3 \ln(2+\sin x) + 3x \frac{\cos x}{2+\sin x}\right) \underbrace{(2+\sin x)^{3x}}_{h(x)}$$

5. (10 points) Find an equation of the tangent line to the graph of the function $f(x) = x + \sqrt{x}$ at $x = 1$.

$$f'(x) = 1 + \frac{1}{2\sqrt{x}}$$

$$f'(1) = 1 + \frac{1}{2} = \frac{3}{2}$$

$$f(1) = 1 + 1 = 2$$

$$\Rightarrow \quad y - 2 = \frac{3}{2}(x-1)$$

$$y = \frac{3}{2}x - \frac{3}{2} + 2 \quad \Rightarrow \boxed{y = \frac{3}{2}x + \frac{1}{2}}$$

6. (10 points) Find the slope of the tangent line to $xy + \cos y + x^2 = 10$ at the point $(3, 0)$.

implicit diff

$$\underbrace{1 \cdot y + x \cdot \frac{dy}{dx}}_{\text{product rule}} + (-\sin y) \cdot \frac{dy}{dx} + 2x = 0$$

@ $x = 3$ & $y = 0$

$$0 + 3\frac{dy}{dx} + 0 + 2 \cdot 3 = 0$$

$$3\frac{dy}{dx} = -6$$

$$\frac{dy}{dx} = -2$$

7. (12 points) Consider a function $f(x)$ whose first and second derivatives are:

$$f'(x) = \frac{x^2-9}{x^2+1} \text{ and } f''(x) = \frac{20x}{(x^2+1)^2}$$

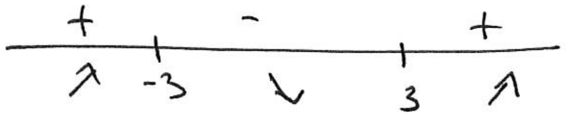
Make sure to justify your answers with a sign diagram or some other work.

(a) (4 points) State the intervals where $f(x)$ is decreasing and those where $f(x)$ is increasing.

$f' = 0 \quad x = \pm 3$

$f' = \emptyset$ N/A

sign of f'



f increasing $(-\infty, -3)$ and $(3, \infty)$

f decreasing $(-3, 3)$

(b) (2 points) Find the x-values of the critical points of $f(x)$ and classify each as a local minimum, local maximum, or neither.

@ $x = -3$ local max

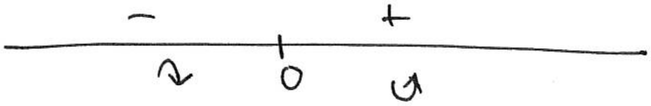
@ $x = +3$ local min

(c) (4 points) State the intervals where $f(x)$ is concave up and those where $f(x)$ is concave down.

$f'' = 0 \quad x = 0$

$f'' = \emptyset$ N/A

sign of f''



concave down $(-\infty, 0)$

concave up $(0, \infty)$

(d) (2 points) Find the x-values of inflection points, if any.

@ $x = 0$ point of inflection

8. (12 points) Consider $f(x) = x^3 - 12x + 4$. Find the absolute minimum and maximum value of f on the interval $[-3, 1]$. Show all your work.

Closed Interval Method

$f'(x) = 3x^2 - 12$

$f' = 0 \quad x^2 = 4$

$x = \pm 2 \Rightarrow x = -2$

$\cancel{x = +2}$ not in $[-3,1]$

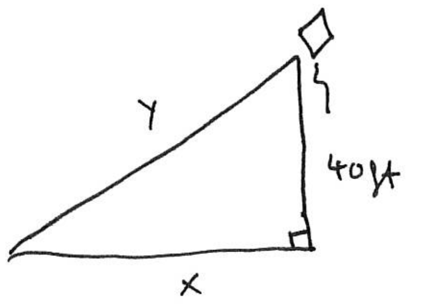
$f' = \emptyset$ N/A

Compare values

x	f(x)
-3	$-27 + 36 + 4 = 13$
-2	$-8 + 24 + 4 = 20$
1	$1 - 12 + 4 = -7$

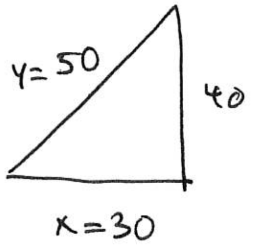
$\boxed{\begin{array}{l} -7 = \text{min value} \\ 20 = \text{max value} \end{array}}$

9. (12 points) A kite 40 ft above the ground moves horizontally at a speed of 10 ft/s. At what rate is the string being released when 50 ft of the string is out? Show all your work.



Given $\frac{dx}{dt} = 10$

Want $\frac{dy}{dt} = ?$ when $y = 50$



$x^2 + (40)^2 = y^2$

$\cancel{2}x \frac{dx}{dt} + 0 = \cancel{2}y \frac{dy}{dt}$

$(30)(10) = (50) \cdot \frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \frac{300}{50} = \boxed{6 \text{ ft/s}}$

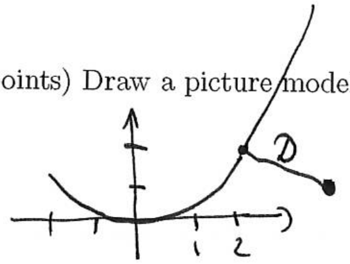
The string is released at a rate of 6 ft/s.

10. (14 points) Find the point(s) on the parabola $y = \frac{1}{2}x^2$ that are closest to the point $(4, 1)$.

(a) (1 point) In words, what quantity (objective) needs to be optimized?

The distance of a point (x,y) on $y=\frac{1}{2}x^2$ to $(4,1)$.

(b) (2 points) Draw a picture modeling this situation.



(c) (4 points) Find a function for the quantity that is being optimized, and find its domain.

Distance $=\sqrt{(4-x)^2+(1-\frac{1}{2}x^2)^2}$

or find minimum (distance)^2.

$f(x)=(4-x)^2+(1-\frac{1}{2}x^2)^2$ Domain $=\mathbb{R}$

(d) (3 points) Find the derivative of your function, and find the critical point(s).

$f'(x)= 2(4-x)(-1) + 2(1-\frac{1}{2}x^2)(-x)$

$= -8 + 2x + (-2x) + x^3$

$= x^3 - 8 = 0 \Rightarrow \boxed{x=2}$

(e) (4 points) Simplify and justify your answer. The final answer is:

sign of f' $\quad \underset{\searrow}{-} \quad \underset{2}{|} \quad \underset{\nearrow}{+}$ $\quad x=2 \Rightarrow y=\frac{1}{2}(2)^2=2$

global min

the point closest to $(4,1)$ is the point $\boxed{(2,2)}$.

11. (6 points) Given $\int_2^3 2f(x)\,dx = 4$ and $\int_3^0 f(x)\,dx = 3$, evaluate $\int_0^2 f(x)\,dx$.

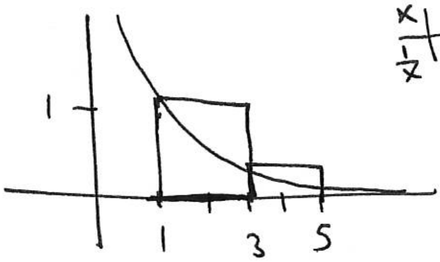
$$\int_0^2 f(x)\,dx = \int_0^3 f(x)\,dx - \int_2^3 f(x)\,dx$$

$$= (-3) - (2)$$

$$= \boxed{-5}$$

12. (8 points) Consider the function $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 5$.

(a) (4 points) Sketch the graph indicating the left hand Riemann Sum L_2 for $n = 2$ and $f(x)$ on the interval $[1, 5]$. Compute its sum. Write your final answer as reduced fraction.



x	1	3	5
$\frac{1}{x}$	1	$\frac{1}{3}$	$\frac{1}{5}$

$n=2$ $\Delta x = 2$

$$L_2 = \left(1 + \frac{1}{3}\right)\cdot 2 = \boxed{\frac{8}{3}}$$

(b) (4 points) Find the actual area on $[1, 5]$. You can leave the answer in the exact form.

$$\int_1^5 \frac{dx}{x} = \ln|x| \Big|_1^5 = \ln 5 - \ln 1 = \boxed{\ln 5}$$

13. (12 points) Find the following definite integrals, no need to simplify your answers.

(a) (6 points) $\int_0^1 (2x^2 - 5x + e^x)\,dx$

$$= \frac{2x^3}{3} - \frac{5x^2}{2} + e^x \Big|_0^1$$

$$= \left(\frac{2}{3} - \frac{5}{2} + e\right) - (0 - 0 + 1)$$

$$= \frac{-11}{6} + e - 1 = e - \frac{17}{6}$$

(b) (6 points) $\int_{-\pi/4}^{\pi/4} (\cos x - 4\sin(2x))\,dx$ $= -\sqrt{2} + 0 = \boxed{-\sqrt{2}}$

$\cos x \rightsquigarrow$ even function

$4\sin(2x) \rightsquigarrow$ odd function $\Rightarrow \int_{-\pi/4}^{\pi/4} 4\sin(2x)\,dx = 0$

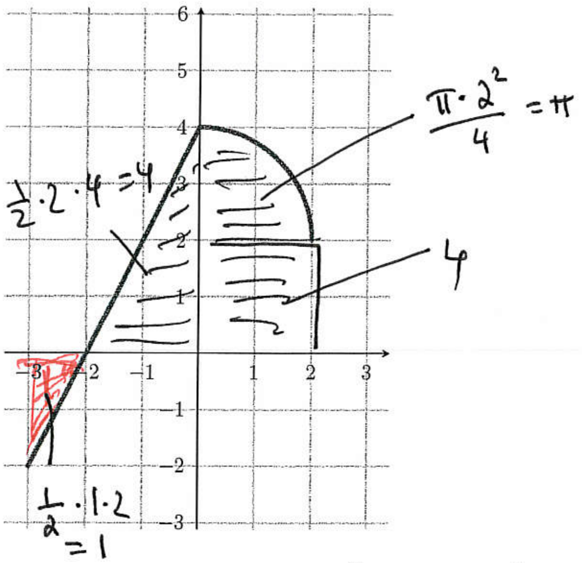
$$= 2 \cdot \int_0^{\pi/4} \cos x \,dx$$

$$= 2 \cdot +\sin x \Big|_0^{\pi/4}$$

$$= 2 \cdot \left(+\frac{\sqrt{2}}{2} \mp 0\right) = +\sqrt{2}$$

14. (14 points) Use the graph of $f(x)$, provided below, to define the area function $A(x) = \int_{-3}^{x} f(x)\,dx$.

(Assume that the graph of f consists of a line segment and a semi-circle)



(a) (5 points) Evaluate $\int_{-3}^{0} f(x)\,dx$, $\int_{0}^{2} f(x)\,dx$, and $\int_{-3}^{2} f(x)\,dx$.

$= -1 + 4$

$= \boxed{3}$

$= \boxed{4+\pi}$

$= 3+4+\pi$

$= \boxed{7+\pi}$

(b) (5 points) Evaluate $A(-3), A(0), A(2)$ and $A'(-1)$.

$A(-3) = 0$

$A(0) = 3$

$A(2) = 7+\pi$

$A'(-1) = 2$ FTC $A'(x) = f(x)$

(c) (4 points) On which intervals is A increasing? decreasing?

A decreasing $(-3,-2)$

increasing $(-2,2)$

15. (18 points) Given $\mathbf{v} = \langle -1, 3\rangle$ and $\mathbf{w} = \langle 4, -2\rangle$.

(a) (4 points) Find the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$. Do not simplify!

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|} = \frac{-4-6}{\sqrt{10}\sqrt{20}} = \frac{-10}{\sqrt{200}} = \frac{-10}{10\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

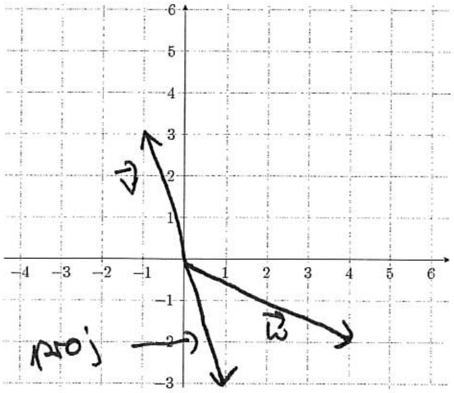
(b) (4 points) Find a unit vector pointing in the same direction of the vector $\mathbf{v}$.

$$\vec{u} = \frac{1}{\sqrt{10}}\langle -1, 3\rangle = \left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right\rangle$$

(c) (4 points) Find $\text{proj}_{\mathbf{v}}\mathbf{w}$.

$$\text{proj}_{\vec{v}}\vec{w} = \frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}\cdot\vec{v} = \frac{-10}{10}\cdot\langle -1, 3\rangle = \langle 1, -3\rangle$$

(d) (6 points) Sketch the vectors $\mathbf{v}$, $\mathbf{w}$ and $\text{proj}_{\mathbf{v}}\mathbf{w}$.



16. (7 points) A tow truck pulls a van along a road. The chain makes an angle of $\pi/6$ with the road and the force of the chain is 500 N. How much work is done in towing the van 20 m?

$$\begin{aligned} \text{Work} &= \text{Force}\cdot\text{Distance} \\ &= \left(500\,N\cdot\cos\pi/6\right)\cdot(20\,m) \\ &= 500\cdot 20\cdot\frac{\sqrt{3}}{2}\ N\cdot m \\ &= 5000\cdot\sqrt{3}\ \text{Joules} \end{aligned}$$

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