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MATH 180 Final Exam
May 4, 2017

Directions. Fill in each of the lines below. Circle your instructor's name and write your TA's name. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR INSTRUCTOR. Good luck!

Print Name: ANSWER KEY

University Email: ______________________

UIN: ______________________

Circle your instructor's name: Shulman Steenbergen Thulin Zhang

TA's Name: ______________________

- VERY IMPORTANT!!! CHECK THAT THE NUMBER AT THE TOP OF EACH PAGE OF YOUR EXAM IS THE SAME. IT IS THE NUMBER PRECEDED BY A POUND (#) SIGN. IF THEY ARE NOT ALL THE SAME, NOTIFY YOUR INSTRUCTOR OR TA RIGHT AWAY.
- All of your work must fit within the boxes on each page for each question. Nothing outside of the box will be graded! If you write outside of the box, there is a good chance that your exam will not be read and therefore not graded.
- A solution for one problem may not go on another page.
- Make clear to the grader what your final answer is.
- Have your student ID ready to be checked when submitting your exam.

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1. (10 points) Find the following derivatives. DO NOT SIMPLIFY YOUR ANSWERS.

(a) $\frac{d}{dx}\left(\frac{\ln x}{3x^2 - x}\right)$

quotient rule

$u = \ln x \qquad v = 3x^2 - x$

$u' = \frac{1}{x} \qquad v' = 6x - 1$

$$= \frac{\left(\frac{1}{x}\right)(3x^2 - x) - (\ln x)(6x-1)}{(3x^2-x)^2}$$

(b) $\frac{d}{dx}\left(x^{3/4} \cdot \cos^{-1}(4x)\right)$

product rule

$u = x^{3/4} \qquad v = \cos^{-1}(4x)$

$u' = \frac{3}{4}x^{-1/4} \qquad v' = \frac{-4}{\sqrt{1-(4x)^2}}$

$$= \left(\frac{3}{4}x^{-1/4}\right)\left(\cos^{-1}(4x)\right) + \frac{-4x^{3/4}}{\sqrt{1-(4x)^2}}$$

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2. (10 points) Find the integrals below.

(a) $\displaystyle\int_0^{\ln(3)} e^{2x}\, dx$ $= \frac{1}{2}e^{2x}\Big|_0^{\ln(3)}$

$= \frac{1}{2}e^{2\ln 3} - \frac{1}{2}e^{0}$

$= \frac{1}{2}e^{\ln 3^2} - \frac{1}{2}$

$= \frac{1}{2}\cdot 9 - \frac{1}{2}$

$= \boxed{4}$

(b) $\displaystyle\int (3x+5)^2\, dx$ $= \int (9x^2+30x+25)\,dx$

$= 3x^3 + 15x^2 + 25x + c$

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3. (10 points) Consider the integral $\int_0^4 (10 - x^2)\, dx$.

(a) Calculate M_2, the midpoint Riemann sum with 2 subintervals. Simplify your answer.

0 1 2 3 4

$\Rightarrow M_2 = (9+1) \cdot 2 = 20$

x	1	3
$f(x)$	9	1

(b) Find the actual value of the integral. Write your answer as a reduced fraction.

$$\left. 10x - \frac{x^3}{3} \right|_0^4$$

$$= \left(10 \cdot 4 - \frac{4^3}{3}\right) - (0 - 0) = 40 - \frac{64}{3} = \frac{120-64}{3} = \boxed{\frac{56}{3}}$$

4. (10 points) Evaluate the following limit. If the limit does not exist, but is $\pm\infty$, tell which it is.

$$\lim_{x \to 0} \frac{e^x - \cos x}{x - \sin x} \quad \text{of type } \frac{0}{0}$$

$$\underset{\text{L'Hôpital}}{=} \lim_{x \to 0} \frac{e^x + \sin x}{1 - \cos x} \quad \text{of type } \frac{1}{0+}$$

Since $\cos x \leq 1 \Rightarrow 1 - \cos x \geq 0 \rightsquigarrow 0^+$

$= \boxed{+\infty}$

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5. (10 points) Let $f(x) = 3x^5 - 20x^3$.

(a) Find and classify each critical point of f as either a local minimum, local maximum, or neither.

$f'(x) = 15x^4 - 60x^2$

$f' = 0$

$15x^2(x^2-4) = 0$

$x = 0$ or $x = \pm 2$

sign of f': $+$ | -2 | $-$ | 0 | $-$ | 2 | $+$

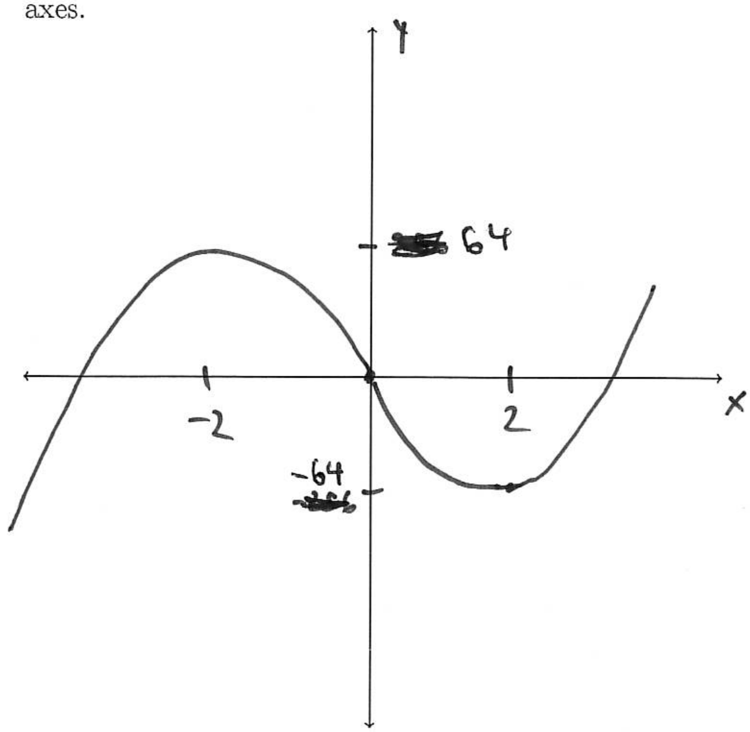
$\boxed{\text{local max } @ x=-2 \quad \text{neither } @ x=0 \quad \text{local min } @ x=2}$

(b) Find the intervals where f is increasing and where f is decreasing.

f increasing on $(-\infty,-2)$ and $(2,\infty)$

f decreasing on $(-2,0) \cup (0,2)$

(c) Using your answers from the parts above, draw a sketch of f on the axes provided. Label your axes.



x	$f(x)$
-2	$-3\cdot 32 + 20\cdot 8 = 64$
0	0
2	

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6. (10 points) Consider the vectors $\mathbf{u} = \langle 2\sqrt{3}, -2\rangle$ and $\mathbf{v} = \langle \sqrt{3}, 1\rangle$.

(a) Find $|\mathbf{u}|$ and $|\mathbf{v}|$.

$$\|\vec{u}\| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{12+4} = \sqrt{16} = 4$$

$$\|\vec{v}\| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

(b) Find $\text{proj}_{\mathbf{v}}\mathbf{u}$.

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\cdot\vec{v} = \frac{2\sqrt{3}\cdot\sqrt{3} + (-2)\cdot 1}{\sqrt{3}\cdot\sqrt{3} + 1\cdot 1}\cdot\langle\sqrt{3}, 1\rangle$$

$$= \frac{4}{4}\langle\sqrt{3}, 1\rangle$$

$$= \langle\sqrt{3}, 1\rangle$$

(c) Find the angle between $\mathbf{u}$ and $\mathbf{v}$. Your answer may not contain $\cos^{-1}$ in it, so simplify it.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} = \frac{4}{4\cdot 2} = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{3}$$

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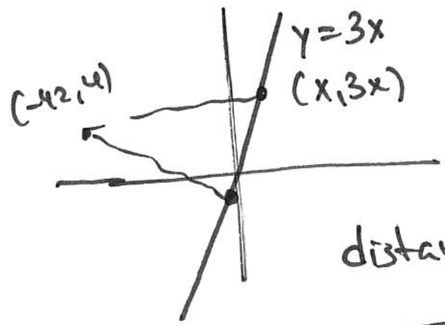
7. (10 points) Find the point on the line $y = 3x$ that is a minimum distance from the point $(-42, 4)$. [Hint: $42^2 = 1764$, although you may not need this.]



distance of $(x, 3x)$ to $(-42, 4)$

$$= \sqrt{(-42-x)^2 + (4-3x)^2}$$

Note that if distance is at a minimum, so is $(\text{distance})^2$.

$$f(x) = (-42-x)^2 + (4-3x)^2 \qquad \text{Domain} = \mathbb{R}$$
$$= (x+42)^2 + (3x-4)^2$$

$$f'(x) = 2(x+42) + 2(3x-4)\cdot 3$$
$$= 2x + 84 + 18x - 24$$
$$= 20x + 60$$

$f' = 0 \quad x = -3$ $\qquad$ sign of f': $-$ on $(-\infty, -3)$ ↘, $+$ on $(-3, \infty)$ ↗

$f' = \emptyset$ N/A

absolute min @ $x = -3$

$y = 3\cdot(-3) = -9$

⇝ $(-3, -9)$ point with min distance

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For each of the multiple choice questions below (Problems 8-13), circle the correct answer. If your answer is not clearly circled, then your answer will be marked wrong. You will be receiving credit for the correct answer as well as for the work you show. So to receive partial credit, show your work or explain you answer in the space provided.

8. (5 points) The function $f(x) = \dfrac{x^2-1}{x^2+1}$ has

(a) no vertical asymptotes and no horizontal asymptotes.
(b) no vertical asymptotes and one horizontal asymptote.
(c) one vertical asymptote and one horizontal asymptote.
(d) two vertical asymptotes and no horizontal asymptotes.
(e) two vertical asymptotes and one horizontal asymptote.

$x^2+1 \neq 0$
$\Rightarrow$ no vertical asymptote

$\lim_{x \to \pm\infty} \frac{x^2-1}{x^2+1} = 1$
$\Rightarrow y=1$ H.A.

9. (5 points) Find $\displaystyle\int_1^2 \frac{x^2-1}{x}\,dx$

$= \int_1^2 (x - \frac{1}{x})dx = \frac{x^2}{2} - \ln|x| \Big|_1^2$

$= (\frac{4}{2} - \ln 2) - (\frac{1}{2} - 0)$

$= \frac{3}{2} - \ln 2$

(a) $\frac{9}{2}$
(b) $\frac{3}{2}$
(c) $\frac{9}{4}$
(d) $\frac{3}{2} - \ln 2$
(e) $5 - \ln 2$

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10. (5 points) Find the slope of the tangent line to the curve $\ln(xy) = x^2 - y^2$ at $(1, 1)$.

(a) $\frac{1}{3}$
(b) 0
(c) -1
(d) $\frac{2}{3}$
(e) 1

implicit diff $(xy)' = 1\cdot y + x\cdot y'$

$$\frac{1}{x\cdot y}\cdot(y + x\cdot y') = 2x - 2yy'$$

@ $(1,1)$

$$\frac{1}{1\cdot 1}\cdot(1 + y') = 2 - 2y'$$

$$3y' = 1$$

$$\boxed{y' = \frac{1}{3}}$$

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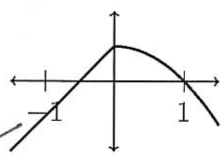
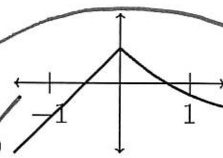
11. (5 points) Which of these graphs satisfies all three of the following conditions? Explain your choice.

(i) $f'(x) > 0$ for $x < 0$ $\Rightarrow$ f increasing on $(-\infty, 0)$
(ii) $f'(x) < 0$ for $x > 0$ $\Rightarrow$ f decreasing on $(0, \infty)$
(iii) $f''(1) > 0$ $\Rightarrow$ f concave up @ $x = 1$



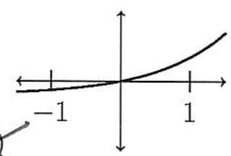
(a)

(i) & (ii) ~~(iii)~~



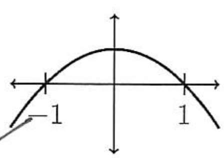
(b)

(i) & (ii) & (iii)



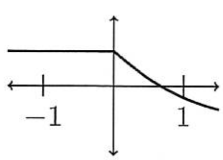
(c)

~~(iii)~~



(d)

(i) & (ii) ~~(iii)~~



(e)

~~(i)~~ (ii) & (iii)

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12. (5 points) Suppose that $\int_1^8 f(x)\,dx = 4$, $\int_5^8 f(x)\,dx = 7$, and $\int_5^1 g(x)\,dx = 3$. Find

$$\int_1^5 (f(x) - 2g(x))\,dx$$

(a) -9
(b) 0
(c) 1
(d) 3
(e) 5

1 — 5 — 8

$= \int_1^5 f(x)dx - 2\cdot\int_1^5 g(x)dx$

$=(4-7)-(-2\cdot 3)$

$= -3 + 6$

$= 3$

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13. (5 points) Find the value of a such that f is continuous for all real numbers.

$$f(x) = \begin{cases} a^{-x} & x \le 1 \\ \dfrac{\sqrt{x^2+3}-2}{x^2-1} & x > 1 \end{cases}$$

(a) 8
(b) 4
(c) 2
(d) 1
(e) such an a does not exist

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} a^{-x} = a^{-1} = \frac{1}{a}$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} \frac{\sqrt{x^2+3}-2}{x^2-1}$ of type $\frac{0}{0}$

$= \lim_{x\to 1} \frac{(\sqrt{x^2+3}-2)(\sqrt{x^2+3}+2)}{(x+1)(x-1)(\sqrt{x^2+3}+2)}$

$= \lim_{x\to 1} \frac{\cancel{x^2+3-4}}{(\cancel{x^2-1})(\sqrt{x^2+3}+2)} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$

$\Rightarrow \frac{1}{a} = \frac{1}{4} \Rightarrow a = 4$