MATH 180 Final Exam
May 9, 2019

Directions. Fill in each of the lines below. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR EXAM PROCTOR. This exam contains 14 pages (including this cover page) and 14 problems. After starting the exam, check to see if any pages are missing. Enter all requested information on this page. You are expected to abide by the University's rules concerning Academic Honesty.

Answers

Name:________________________

UIN:________________________

UIC Email:________________________

Signature: ________________________

The following rules apply:

- You may *not* use your books, notes, calculators, or any electronic device including cell phones. Only pencils/pens allowed.
- You must show all of your work. An answer, right or wrong, without the proper justification will receive little to no credit.
- You *must* complete your work in the space provided. We will be scanning your answers into our grading system, so any work you do that is out of place, too close to the page border, or on the wrong page will *not* be graded!

Circle your instructor.

- Jenny Ross
- Mercer (Tabes) Bridges
- Matthew Lee

Write your Initials (First, Middle, Last)____________

1. (14 points) Differentiate the following functions, using logarithmic differentiation if needed. Do not simplify your answers.

   (a) (4 points) $f(x) = \dfrac{\arcsin(x+5)}{e^x}$

quotient rule $u = \arcsin(x+5)$ $v = e^x$

$u' = \dfrac{1}{\sqrt{1-(x+5)^2}} \cdot 1$ $v' = e^x$

$$f'(x) = \frac{\dfrac{e^x}{\sqrt{1-(x+5)^2}} - e^x \arcsin(x+5)}{e^{2x}}$$

   (b) (4 points) $h(x) = \tan(2^x)\ln(x)$

product rule $u = \tan(2^x)$ $v = \ln x$

$u' = \sec^2(2^x) \cdot \ln 2 \cdot 2^x$ $v' = \frac{1}{x}$

$$h'(x) = \sec^2(2^x) \ln 2 \cdot 2^x \cdot \ln x + \tan(2^x) \cdot \frac{1}{x}$$

   (c) (6 points) $g(x) = (x-5)^{\sin(x)}$

use log differentiation

$$\ln g(x) = \sin x \cdot \ln(x-5)$$

$$\frac{1}{g(x)} \cdot g'(x) = \cos x \cdot \ln(x-5) + \sin x \cdot \frac{1}{x-5}$$

$$g'(x) = \left(\cos x \cdot \ln(x-5) + \sin x \cdot \frac{1}{x-5}\right) \cdot (x-5)^{\sin x}$$

Write your Initials (First, Middle, Last)____________

2. (14 points) Evaluate the following integrals:

(a) (4 points) $\displaystyle\int \left(\frac{1}{1+x^2} - \cos(x)\right)\, dx$

$= \arctan x - \sin x + c$

(b) (5 points) $\displaystyle\int_{-2}^{0} (e^{3x} + 5x)\, dx$

$= \left(\frac{1}{3}e^{3x} + \frac{5}{2}x^2\right)\Big|_{-2}^{0}$

$= \left(\frac{1}{3}e^{0} + 0\right) - \left(\frac{1}{3}e^{-6} + \frac{5}{2}(-2)^2\right)$

$= \frac{1}{3} - \frac{1}{3e^6} - 10 = \frac{-29}{3} - \frac{1}{3e^6}$

(c) (5 points) $\displaystyle\int_{-3}^{3} (\sin^3(x))\, dx = 0$

b/c $\sin^3 x$ is an odd function

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3. (14 points) Compute the following limits, using L'Hôpital's Rule if needed. If a limit is $+\infty$ or $-\infty$ state which one, and if the limit does not exist, explain why.

(a) (3 points) $\lim_{x\to -2} \dfrac{3x^2+5}{x^3-5x^2+8}$ $= \dfrac{12+5}{-8-20+8} = \dfrac{17}{-20} = -\dfrac{17}{20}$

(b) (3 points) $\lim_{x\to\infty} \dfrac{x^2-5x+4}{x^3-4x+2}$ $= \lim \dfrac{\frac{1}{x} - \frac{5}{x^2} + \frac{4}{x^3}}{1 - \frac{4}{x} + \frac{2}{x^3}} = \dfrac{0}{1} = 0$

or use L'Hôpital (of type $\frac{\infty}{\infty}$)

$= \lim_{x\to\infty} \dfrac{2x-5}{3x^2-4}$ of type $\frac{\infty}{\infty}$

$\underset{L'H}{=} \lim \dfrac{2}{6x} = 0$

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Number 3 (continued)

(c) (3 points) $\lim_{x \to 3} \frac{x^2 - 9}{x^2 - 6x + 9}$ of type $\frac{0}{0}$

$$\underset{\text{L'Hôpital}}{=} \lim_{x \to 3} \frac{2x}{2x-6} \text{ of type } \frac{6}{0}$$

since as $x \to 3$ $2x-6 \to \pm 0$

this limit would be $\pm\infty$ and

thus does not exist

or $\lim_{x \to 3^-} f(x) = -\infty$

$\lim_{x \to 3^+} f(x) = +\infty$

$\Rightarrow \lim_{x \to 3} f(x)$ DNE

(d) (5 points) $L = \lim_{x \to 0+} (1 - x)^{5/x}$ use natural log

$$\ln L = \lim_{x \to 0^+} \frac{5}{x} \ln(1-x)$$

$$= \lim_{x \to 0^+} \frac{5\ln(1-x)}{x} \text{ of type } \frac{0}{0}$$

$$\underset{\text{L'Hôpital}}{=} \lim_{x \to 0^+} \frac{\frac{5}{1-x}(-1)}{1} = -5$$

$$\Rightarrow \quad L = e^{-5}$$

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4. (10 points) Find the slope of the tangent line to $3x^2y - y^3 - x = 9$ at the point $(2, 1)$

use implicit differentiation

$$6xy + 3x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} - 1 = 0$$

$$@ (2,1) \Rightarrow x=2 \; \& \; y=1$$

$$6(2)(1) + 3(2)^2\frac{dy}{dx} - 3(1)^2\frac{dy}{dx} - 1 = 0$$

$$12 + 12\frac{dy}{dx} - 3\frac{dy}{dx} - 1 = 0$$

$$9\frac{dy}{dx} = -11$$

$$\frac{dy}{dx} = -\frac{11}{9} \leftarrow \text{slope}$$

5. (9 points) Find the absolute maximum/minimum of $f(x) = 2x^3 - 3x^2 - 12x + 5$ on the interval $[0, 3]$.

$$f'(x) = 6x^2 - 6x - 12$$
$$= 6(x^2 - x - 2)$$
$$= 6(x-2)(x+1)$$

(i) $f' = 0 \Rightarrow \cancel{x = -1}$ & $x = 2$

not in [0, 3]

(ii) $f' = \emptyset$ N/A

(iii) endpoints $x = 0$ & $x = 3$

Compare points of interest:

x	$f(x)$
0	$\boxed{5 = \text{max value}}$
2	$\boxed{-15 = \text{min value}}$
3	$-4 \leftarrow$

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6. (10 points) Use the **limit definition** of the derivative to find the derivative of $f(x) = 3x^2 - 1$. No other methods will receive points.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(3(x+h)^2 - 1) - (3x^2 - 1)}{h}$$

$$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) \cancel{-1} - 3x^2 \cancel{+1}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 \cancel{- 3x^2}}{h}$$

$$= \lim_{h \to 0} 6x + 3h = \boxed{6x}$$

7. (9 points) Find the linear approximation to $f(x) = -3x^2 + 5x - 8$ at $x = 2$ and use it to approximate the value of $f(x)$ at $x = 2.1$. Is this an over estimation or under estimation? Justify by referencing concavity.

$a = 2 \Rightarrow L(x) = f(2) + f'(2)(x-2)$

$f(2) = -10$

$f'(x) = -6x + 5$

$f'(2) = -7$

$\Rightarrow L(x) = -10 + (-7)(x-2)$

$f''(x) = -6 < 0 \Rightarrow$ graph of f is concave down

OVER ESTIMATE

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8. (10 points) Let $\overrightarrow{w} = \langle -3, 4 \rangle$ and $\overrightarrow{v} = \langle -2, 1 \rangle$

   (a) (2 points) Simplify $-\overrightarrow{w} + 5\overrightarrow{v}$

   $$= \langle +3 + 5(-2), -4 + 5 \cdot 1 \rangle$$

   $$= \langle -7, 1 \rangle$$

   (b) (2 point) Find the magnitude of $\overrightarrow{v}$

   $$\|\vec{v}\| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$$

   (c) (2 points) Find the value of $\overrightarrow{w} \cdot \overrightarrow{v}$

   $$= (-3)(-2) + (4)(1)$$

   $$= 6 + 4 = 10$$

   (d) (2 points) Find a unit vector parallel to $\overrightarrow{w}$

   $$\|\vec{w}\| = \sqrt{(-3)^2 + (4)^2} = \sqrt{25} = 5$$

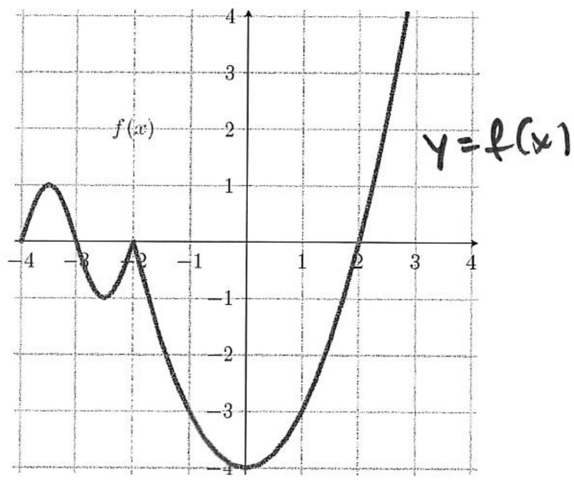
   $$\Rightarrow \vec{u} = \frac{1}{5} \cdot \langle -3, 4 \rangle = \left\langle \frac{-3}{5}, \frac{4}{5} \right\rangle$$

   (e) (2 points) Find the projection of $\overrightarrow{w}$ onto $\overrightarrow{v}$

   $$\text{proj}_{\vec{v}} \vec{w} = \frac{\vec{v} \cdot \vec{w}}{\vec{v} \cdot \vec{v}} \cdot \vec{v} = \frac{10}{5} \cdot \langle -2, 1 \rangle$$

   $$= \langle -4, 2 \rangle$$

9. (7 points) For the graph of $f(x)$ below find:



(a) (2 points) State on which intervals $f'(x) < 0$ by writing your answer in interval notation, using decimals if necessary.

f is decreasing & $f' < 0$ on the intervals
$(-3.5, -2.5)$ and $(-2, 0)$

(b) (3 points) State on which intervals $f(x)$ is concave up/concave down by writing your answer in interval notation, using decimals if necessary.

f concave up on $(-3, -2)$ and $(-2, 3)$

f concave down on $(-4, -3)$

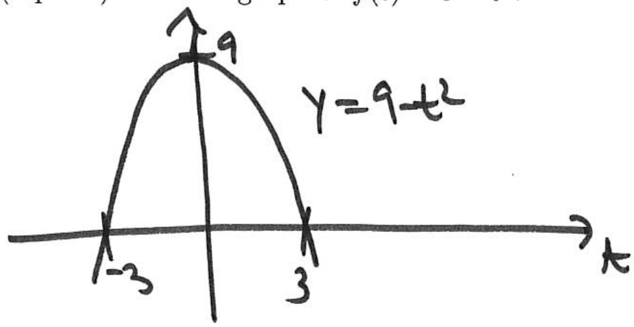
(c) (2 points) Identify any inflection points and write your answer in a point format.

Only one point of inflection at $x = -3$ or
at the point $(-3, 0)$.

Write your Initials (First, Middle, Last)__________

10. (5 points) Given $A(x) = \int_{-3}^{x} (9 - t^2)\, dt$ on $[-3, 4]$, answer the following:

a. (2 point) Sketch a graph of $f(t) = 9 - t^2$.



b. (2 point) When does $A(t)$ reach its maximum value? Justify your answer.

at $x=3$ because up until $x=3$ $A(x)$ is adding area, after $x=3$ the area gets subtracted.

c. (2 point) Find the following: $A(2) = \int_{-3}^{2} (9-t^2)\,dt$

$$= \left(9t - \frac{t^3}{3}\right)\Big|_{-3}^{2}$$

$$= \left(9 \cdot 2 - \frac{8}{3}\right) - \left(-27 + \frac{27}{3}\right) = \left(18 - \frac{8}{3}\right) + 27 - \frac{27}{3}$$

$$= 45 - \frac{35}{3} = \frac{135 - 35}{3} = \frac{100}{3}$$

d. (2 points) Find the following: $A'(x)$ and $A''(x)$.

$A'(x) = 9 - x^2$ Fundamental Theorem of Calculus

$A''(x) = -2x$

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11. (4 points) (a) (2 points) Find the average value of $f(x) = -2x + 5$ on $[-2, 2]$.

$$\text{ave value} = \frac{1}{2-(-2)} \int_{-2}^{2} (-2x+5)\,dx$$

$$= \frac{1}{4}\left(-x^2+5x\right)\Big|_{-2}^{2} = \frac{1}{4}(-4+10) - \frac{1}{4}(-4-10)$$

$$= \frac{1}{4}(\cancel{-4}+10+\cancel{4}+10) = \frac{20}{4} = \boxed{5}$$

(b) (2 points) Find the c on $[-2, 2]$ which satisfies the Mean Value Theorem for integrals.

by MVT there is a c with $f(c) = 5$

$$-2c+5 = 5$$

$$\boxed{c=0}$$

12. (6 points) In an attempt to be lazy, Luke Skywalker uses the force to pull a box along the ground. Luke pulls the box from the couch at an angle of incline of $\frac{\pi}{4}$ with a force of 200 N. This force moved the box 100 meters. How much "work" is done by Luke?

Work = Force · Distance

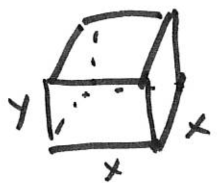
$= 200\text{ N} \cdot \cos \pi/4 \cdot 100\text{ m}$

$= 200 \cdot \frac{\sqrt{2}}{2} \cdot 100$ Joules

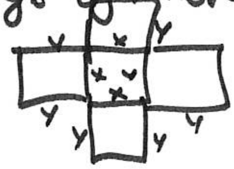
$= 10{,}000 \cdot \sqrt{2}$ Joules

Write your Initials (First, Middle, Last)____________

13. (9 points) What are the dimensions of a box, with an **open top** and square base, of maximum volume that can be constructed using 300 square feet of material.



Find max volume subject to 300ft of material



Find max volume $= x^2 \cdot y$ subject to

$$x^2 + 4xy = 300$$

$$4xy = 300 - x^2$$

$$y = \frac{300 - x^2}{4x}$$

$\Rightarrow f(x) = x^2 \cdot \frac{300 - x^2}{4x}$ volume

$x \geq 0$

$y \geq 0$ $y = 0 \rightsquigarrow x^2 = 300$

$x = \sqrt{300}$

$= \frac{1}{4} x \cdot (300 - x^2)$

$f(x) = 75x - \frac{1}{4}x^3$

Domain = ?

$= [0, \sqrt{300}]$ or $(0, \sqrt{300})$

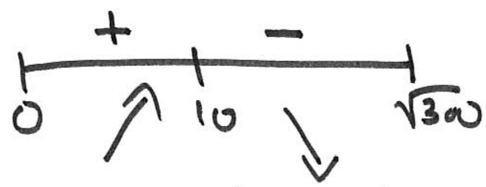
$f'(x) = 75 - \frac{3}{4}x^2 = 0$

$\frac{3}{4}x^2 = 75$

$x^2 = \cancel{75}^{25} \cdot \frac{4}{\cancel{3}} = 100$

$x = 10$

sign of f'



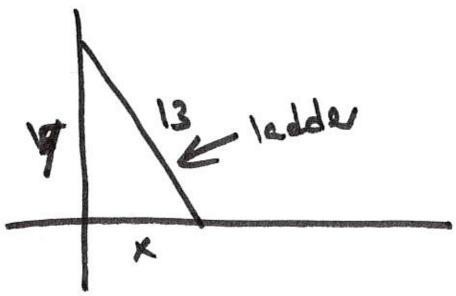
max volume when $x = 10$

$$y = \frac{300 - (10)^2}{4 \cdot 10} = \frac{200}{40} = 5$$

Dimensions are: $10_{ft} \times 10_{ft} \times 5_{ft}$

Write your Initials (First, Middle, Last)____________

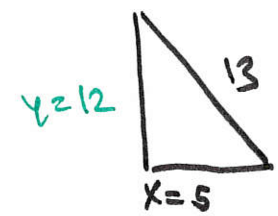
14. (10 points) A ladder, 13 feet in length, is leaning against a wall and starts sliding down the wall at a rate of 1 foot per second. How quickly is the base of the ladder sliding away from the wall when the bottom of the ladder is 5 feet away from the wall?



Given: $\frac{dy}{dt} = -1$ ft/s

Want $\frac{dx}{dt} = ?$ when $x=5$

$x^2 + y^2 = (13)^2$



$5^2 + y^2 = (13)^2$

$y^2 = 169 - 25 = 144$

$y = 12$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

or

$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

$5 \cdot \frac{dx}{dt} + (12)(-1) = 0$

$\frac{dx}{dt} = \frac{-12}{5}$ the bottom of the ladder is sliding away from the wall at a rate of $\frac{12}{5}$ ft/s

This page can be used as scratch paper. It WILL NOT BE GRADED, so please SHOW YOUR WORK WITH YOUR PROBLEMS.