Math 180
Spring 2014
First Midterm
9/24/2014
Time Limit: 2 Hours

Name (Print): ______________________________

This exam contains 11 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

The following rules apply:

- You are expected to abide by the University's rules concerning Academic Honesty.
- You may *not* use your books, notes, or any electronic device including cell phones.
- You must show all of your work. An answer, right or wrong, without the proper justification will receive little to no credit.
- You must complete your work in the space provided. No extra paper will be provided.

Circle your instructor.

- Adrovic @ 11am
- Adrovic @ 1pm
- Cabrera
- Dumas
- Kashcheyeva
- Kobotis
- Lowman
- Shin @ 10am
- Shin @ 2pm
- Shulman @ 8am
- Shulman @ 9am

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) Using the definition of the derivative as the limit of a difference quotient, find $\dfrac{dy}{dx}$ if $y = 2x^3 - 3x^2$.

SOLUTION:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{[2(x+h)^3 - 3(x+h)^2] - [2x^3 - 3x^2]}{h} \\
&= \lim_{h\to 0} \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - 3x^2 - 6xh - 3h^2 - 2x^3 + 3x^2}{h} \\
&= \lim_{h\to 0} \left(6x^2 + 6xh + 2h^2 - 6x - 3h\right) = 6x^2 - 6x
\end{aligned}$$

GRADING RUBRIC:

0 points – If the student does not use the definition of the derivative

2 points – If the student properly sets up the limit definition using $y = 2x^3 - 3x^2$, but does not continue correctly from here

4 points – If the student properly multiplies out either $2(x+h)^3$ or $3(x+h)^2$, but not both

6 points – If the student properly multiplies out both $2(x+h)^3$ and $3(x+h)^2$

8 points – If the student properly applies algebra to reach the step $\lim\limits_{h\to 0} \left(6x^2 + 6xh + 2h^2 - 6x - 3h\right)$

10 points – Correct calculation of the limit to find $\dfrac{dy}{dx} = 6x^2 - 6x$

-2 points – If a student makes a small algebra error

-2 points – If the student does not write a limit

2. (10 points) Calculate the following limit or justify why it does not exist. $\lim_{x\to 3}\sqrt{\dfrac{7e^{3-x}+x^2}{\cos(\pi x)+2}}$

SOLUTION:

$$\lim_{x\to 3}\sqrt{\frac{7e^{3-x}+x^2}{\cos(\pi x)+2}}=\sqrt{\frac{7e^{0}+3^2}{\cos(3\pi)+2}}=\sqrt{\frac{16}{1}}=4$$

GRADING RUBRIC:

6 points – If the student tries to plug in the value, but does no simplifying

10 points – If the student properly simplifies to the limit of 4

-1 point – If a student does not simplify $\cos(3\pi)$ or makes an arithmetic mistake

-1 point – If $\lim_{x\to 3}$ is still present after plugging in -3

3. (10 points) Let $s(t) = 2t^3 - 15t^2 + 29t - 10$ denote the position function of an object where s is measured in feet and t is measured in seconds. Find all values of t where the instantaneous velocity of the object is 5 feet per second.

Solution: The instantaneous velocity is $s'(t) = 6t^2 - 30t + 29$. Then

$$\begin{aligned} 6t^2 - 30t + 29 &= 5 \\ 6t^2 - 30t + 24 &= 0 \\ t^2 - 5t + 4 &= 0 \\ (t-1)(t-4) &= 0 \\ t &= 1, 4 \end{aligned}$$

Grading Rubric:

2 points – If the student found correct parts of $s'(t)$, but not the whole derivative

5 points – If the student found $s'(t)$ correctly

2 points – If the student knows to set $s'(t) = 5$

5 points – If the student properly found $s'(t)$, knew to set $s'(t) = 5$, and found $t = 1, 4$

4. (10 points) Evaluate the following limit or justify why it does not exist. $\displaystyle\lim_{x\to+\infty}\frac{\sin^2 x-2x^3}{5x^3+2}$

SOLUTION:

$$\lim_{x\to+\infty}\frac{\sin^2 x-2x^3}{5x^3+2} = \lim_{x\to+\infty}\frac{\sin^2 x}{5x^3+2}-\lim_{x\to+\infty}\frac{2x^3}{5x^3+2}$$

We know $0\le\sin^2 x\le 1$ so for $x>0$,

$$0\le\frac{\sin^2 x}{5x^3+2}\le\frac{1}{5x^3+2}$$

Then

$$\lim_{x\to+\infty}0 = 0$$
$$\lim_{x\to+\infty}\frac{1}{5x^3+2} = \lim_{x\to+\infty}\frac{\frac{1}{x^3}}{5+\frac{2}{x^3}}=\frac{0}{5}=0.$$

By the Squeeze Theorem

$$\lim_{x\to+\infty}\frac{\sin^2 x}{5x^3+2}=0.$$

Now

$$\lim_{x\to+\infty}\frac{2x^3}{5x^3+2}=\lim_{x\to+\infty}\frac{2}{5+\frac{2}{x^3}}=\frac{2}{5}$$

Therefore

$$\lim_{x\to+\infty}\frac{\sin^2 x-2x^3}{5x^3+2}=\lim_{x\to+\infty}\frac{\sin^2 x}{5x^3+2}-\lim_{x\to+\infty}\frac{2x^3}{5x^3+2}=0-\frac{2}{5}=-\frac{2}{5}$$

GRADING RUBRIC:

4 points – If the student correctly divide every term by x^3

4 points – If the student justify why $\displaystyle\lim_{x\to+\infty}\frac{\sin^2 x}{x^3}=0$. [Note: Proper justification includes using the Squeeze Theorem, or writing an explanation that $\sin^2 x$ is bounded while x^3 goes to $+\infty$. Simply crossing out the term and saying it is 0 received no credit.]

2 points – If the student evaluate the full limit to be $-\frac{2}{5}$

Note: If the student simply said the leading coefficients were -2 and 5 so the limit is $-\frac{2}{5}$ received no credit.

5. (10 points) Let $f(x) = \begin{cases} x^2 + ax + 1 & \text{if } x \leq 2 \\ a & \text{if } x > 2 \end{cases}$.

   (a) (5 points) Find a value for a such that f is continuous at $x = 2$.

   (b) (5 points) For the value of a you found in part (a), is $f(x)$ differentiable at $x = 2$? Justify your answer.

SOLUTION: (a)

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (x^2 + ax + 1) = 2^2 + 2a + 1 = 5 + 2a \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} a = a \end{aligned}$$

Therefore

$$\begin{aligned} 5 + 2a &= a \\ a &= -5. \end{aligned}$$

(b) If $a = -5$, then for $x < 2$, $f'(x) = 2x - 5$. So coming from the left $f'(2) = 2(2) - 5 = -1$. For $x > 2$, $f(x) = -5$, so $f'(x) = 0$. So coming from the right $f'(2) = 0$. Since the derivative from the left does not match the derivative from the right, f is not differentiable at $x = 2$.

GRADING RUBRIC: (a)

2 points total – If the student sets up $5 + 2a = a$ without limits

2 points – If the student properly found the left-hand limit

2 points – If the student properly found the right-hand limit

5 points – Only if the student properly found the right- and left-hand limits and found $a = -5$

(b)

2 points – If the student properly found the derivative from the left

2 points – If the student properly found the derivative from the right

5 points – Only if the student found the derivatives from the left and right and concluded that since they are not equal, f is not differentiable at $x = 2$

6. (10 points) Calculate the following limit or justify why it does not exist. $\lim_{x\to 1} \frac{1-x}{1-\sqrt{x}}$

SOLUTION:

$$\begin{aligned} \lim_{x\to 1} \frac{1-x}{1-\sqrt{x}} &= \lim_{x\to 1} \frac{1-x}{1-\sqrt{x}} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}} \\ &= \lim_{x\to 1} \frac{(1-x)(1+\sqrt{x})}{1-x} \\ &= \lim_{x\to 1} (1+\sqrt{x}) = 2 \end{aligned}$$

GRADING RUBRIC:

0 points – If the student uses L'Hopital's Rule

2 points – If the student got a $\frac{0}{0}$ limit by plugging in $x = 1$, but did nothing else

4 points – If the student knew to multiply numerator and denominator by $1+\sqrt{x}$, but did not do it correctly; this includes not canceling $\frac{1-x}{1-x}$

7 points – If the student properly rationalized and got to $\lim_{x\to 1}(1+\sqrt{x})$, but did not evaluate the limit correctly

10 points – If the student properly rationalized and got the correct limit

-3 points – If the student did not write $\lim_{x\to 1}$ anywhere

7. (10 points) Write the equation of the tangent line to $y = x^2 - 3x + 1$ at $x = -1$.

SOLUTION: A point on the line is $(-1, y(-1)) = (-1, 5)$. Since $y' = 2x - 3$, the slope of the tangent line it $y'(-1) = -5$. The equation of the tangent line is then

$$y - 5 = -5(x + 1).$$

GRADING RUBRIC:

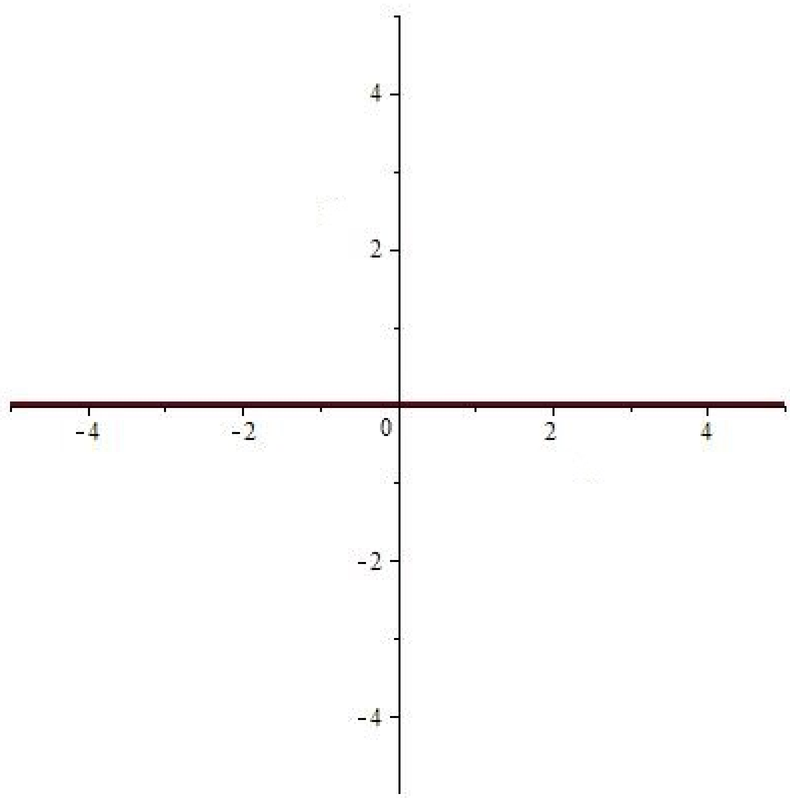
2 points – If the student properly found the point $(-1, 5)$ on the tangent line

3 points – If the student properly found y'

5 points – If the student properly found y' and the slope $y'(-1) = -5$

10 points – Only if the student found the point and the slope correctly and wrote the equation of the tangent line.

8. (10 points) Draw a graph of one function, $f(x)$, that satisfies the following two properties:
   (i) $f'(2) = 0$ and (ii) $f'(0)$ does not exist.



Grading Rubric:

5 points – If the student has a graph that properly displays a *function* where $f'(2) = 0$ or $f'(0)$ does not exist, but not both

10 points – If the student has a correct graph of a *function* with both properties satisfied

9. (10 points) Find the following derivatives. You do not need to simplify your final answer.

   (a) (5 points) $\dfrac{d}{dt}\left(\dfrac{t^2+\cos t}{1-\tan t}\right)$.

   (b) (5 points) $\dfrac{d}{dx}\left(e^x \cdot x^{-3}\right)$.

SOLUTION: (a)

$$\frac{d}{dt}\left(\frac{t^2+\cos t}{1-\tan t}\right) = \frac{(1-\tan t)(2t-\sin t)-(t^2+\cos t)(-\sec^2 t)}{(1-\tan t)^2}$$

(b)

$$\frac{d}{dx}\left(e^x \cdot x^{-3}\right) = e^x \cdot x^{-3} + e^x \cdot (-3x^{-4})$$

GRADING RUBRIC: (a)

If an attempt is made to use the quotient rule, then:

1 point – If the student correctly set up the quotient rule

1 point – If the student properly found the derivative of the numerator

1 point – If the student properly found the derivative of the denominator

2 points – Only if the student properly found the derivative of the numerator and denominator and applied the quotient rule correctly

-1 point – If the student made a simplification error or a minor algebra error

If no attempt was made to use the quotient rule (note that f'/g' does not qualify as an attempt), then:

0 points – If the student did not attempt the quotient rule

(b)

If an attempt is made to use the product rule, then:

1 point – If the student correctly set up the product rule

1 point – If the student properly found the derivative of e^x

1 point – If the student properly found the derivative of the x^{-3}

2 points – Only if the student properly found the derivative of both factors and applied the product rule correctly

-1 point – If the student made a simplification error or a minor algebra error

If no attempt was made to use the product rule (note that $f' \cdot g'$ does not qualify as an attempt), then:

0 points – If the student did not attempt the product rule

OR

If the student rewrote the function as $\dfrac{e^x}{x^3}$ and applied the quotient rule, then the rubric for part (a) was used

10. (10 points) Let $f(x) = \dfrac{x^2+1}{x^2-4}$. Identify all of the vertical asymptotes of $y = f(x)$ and compute the one-sided limits for each.

SOLUTION: Since the denominator factors as $(x-2)(x+2)$, the only possible vertical asymptotes are $x = 2$ and $x = -2$.

$$\begin{aligned}
\lim_{x\to 2^-} \frac{x^2+1}{x^2-4} &= \frac{\text{approaches } 5}{\text{negative and approaches } 0} = -\infty \\
\lim_{x\to 2^+} \frac{x^2+1}{x^2-4} &= \frac{\text{approaches } 5}{\text{positive and approaches } 0} = +\infty \\
\lim_{x\to -2^-} \frac{x^2+1}{x^2-4} &= \frac{\text{approaches } 5}{\text{positive and approaches } 0} = +\infty \\
\lim_{x\to -2^+} \frac{x^2+1}{x^2-4} &= \frac{\text{approaches } 5}{\text{negative and approaches } 0} = -\infty
\end{aligned}$$

Therefore both $x = 2$ and $x = -2$ are vertical asymptotes.

GRADING RUBRIC:

0 points – If the student found $x = \pm 2$, but got finite limits for the one-sided limits

0 points – If the student found $x = \pm 2$, but did not justify that they are vertical asymptotes through limits

2 points – If the student found $x = \pm 2$, and computed some limits, but the limit calculations were incorrect

2 points each – If the student found the one-sided limit correctly