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MATH 180 Exam 2
October 25, 2016

Directions. Fill in each of the lines below. Circle your instructor's name and write your TA's name. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR INSTRUCTOR. Good luck!

Print Name: ____________________

University Email: ____________________

UIN: ____________________

Circle your instructor's name: Boester Dumas Embers Riedl Shulman

TA's Name: ____________________

- VERY IMPORTANT!!! CHECK THAT THE NUMBER AT THE TOP OF EACH PAGE OF YOUR EXAM IS THE SAME. IT IS THE NUMBER PRECEDED BY A POUND (#) SIGN. IF THEY ARE NOT ALL THE SAME, NOTIFY YOUR INSTRUCTOR OR TA RIGHT AWAY.
- All of your work must fit within the boxes on each page for each question. Nothing outside of the box will be graded! If you write outside of the box, there is a good chance that your exam will not be read and therefore not graded.
- A solution for one problem may not go on another page.
- Make clear to the grader what your final answer is.
- Have your student ID ready to be checked when submitting your exam.

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1. (10 points) Consider the curve defined by $x^2 - y^2 + 2y^3 = 10$. Calculate $\frac{dy}{dx}$ at the point $(3, 1)$.

implicit diff

$$2x - 2y\frac{dy}{dx} + 6y^2\frac{dy}{dx} = 0$$

@ (3,1) plug in x=3 & y=1

$$6 - 2\frac{dy}{dx} + 6\frac{dy}{dx} = 0$$

$$4\frac{dy}{dx} = -6$$

$$\frac{dy}{dx} = -\frac{6}{4} = \boxed{-\frac{3}{2}}$$

2. (10 points) Find the equation of the tangent line to $y = \arcsin\left(\frac{1}{x}\right)$ at $x = 2$.

$$y' = \frac{1}{\sqrt{1-(\frac{1}{x})^2}} \cdot \left(-\frac{1}{x^2}\right)$$

$$y'(2) = \frac{1}{\sqrt{1-\frac{1}{4}}} \cdot \left(-\frac{1}{4}\right)$$

$$= \frac{1}{\sqrt{\frac{3}{4}}} \cdot \left(-\frac{1}{4}\right)$$

$$= \frac{2}{\sqrt{3}} \cdot \left(-\frac{1}{4}\right) = \frac{-1}{2\sqrt{3}}$$

$y(2) = \arcsin\left(\frac{1}{2}\right)$

$\sin(\pi/6) = \frac{1}{2}$

$\Rightarrow \arcsin(\frac{1}{2}) = \pi/6$

$$\Rightarrow y - \pi/6 = -\frac{1}{2\sqrt{3}}(x-2)$$

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3. (18 points) Find the following derivatives. DO NOT SIMPLIFY YOUR ANSWERS.

(a) $\frac{d}{dx}\left((\ln x)^3\right) = 3(\ln x)^2 \cdot \frac{1}{x}$

(b) $\frac{d}{dx}\left(\arctan(e^x - x)\right) = \frac{1}{1+(e^x-x)^2} \cdot (e^x - 1)$

(c) $\frac{d}{dx}\left(2^{-x} x^{\sqrt{5}}\right) = (\ln 2)\, 2^{-x}(-1)\, x^{\sqrt{5}} + 2^{-x} \sqrt{5}\, x^{\sqrt{5}-1}$

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4. (10 points) The function $y = x^{(x^2)}$ has one critical point in the interval $(0, \infty)$. Find this critical point. Note that you are NOT required to determine whether this point is a local extremum.

$y' = ?$ use logarithmic differentiation

$y = x^{x^2}$

$\ln y = x^2 \cdot \ln x$

$\frac{1}{y} \cdot y' = 2x \ln x + x^2 \cdot \frac{1}{x}$

$y' = (2x \ln x + x) \cdot x^{x^2}$

$y' = 0 \Rightarrow 2x \ln x + x = 0$

$x(2\ln x + 1) = 0$

$\cancel{x=0}$ or $2\ln x + 1 = 0$

$\ln x = -\frac{1}{2} \Rightarrow \boxed{x = e^{-1/2}}$

5. (10 points) Find the absolute maximum and absolute minimum of $\frac{x+1}{(x+2)^2}$ on the interval $[-1, 2]$. Determine both the extreme value y and the location x where it occurs.

$f = \frac{x+1}{(x+2)^2}$

$u = x+1 \quad v = (x+2)^2$

$u' = 1 \quad v' = 2(x+2)$

$f' = \frac{(x+2)^2 - (x+1)2(x+2)}{(x+2)^2}$

$= \frac{(x+2)((x+2) - 2(x+1))}{(x+2)^2}$

$= \frac{(x+2)(-x)}{(x+2)^2}$

(i) $f' = 0 \Rightarrow x+2 = 0$ or $x = 0$

$x = -2$ not in $[-1,2] \Rightarrow \boxed{x=0}$

(ii) $f' = \emptyset \quad x = -2$ not in $[-1,2]$

(iii) end points $\boxed{x=-1}$ & $\boxed{x=2}$

Compare values

x	
-1	0 = min value @ $x=-1$
0	$\frac{1}{4}$ = max value @ $x=0$
2	$\frac{3}{16}$

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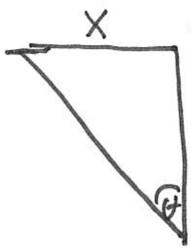
6. (12 points) You are watching a friend run the Chicago marathon. You are standing on the finish line, 20 feet from the point where your friend will cross it. As your friend approaches the finish line at a speed of 12 feet per second, you record a video of their final leg of the race with your cell phone, turning the phone as needed to keep your friend centered in the image. At the moment your friend is 15 feet from the finish line, at what rate must you be turning your phone (in radians per second) to keep them exactly centered in the frame?

DRAW A FIGURE showing the situation, labeling each variable you use in the problem.

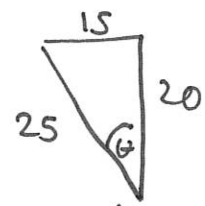


↙ finish line

x = distance to finish line

Know : $\frac{dx}{dt} = -12$ ft/sec

Want : $\frac{d\theta}{dt}$ @ instant $x = 15$



$\sec\theta = \frac{hyp}{adj} = \frac{25}{20} = \frac{5}{4}$

$\tan\theta = \frac{x}{20}$

$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt}$

$\left(\frac{5}{4}\right)^2 \cdot \frac{d\theta}{dt} = \frac{1}{20}(-12)$

$\frac{d\theta}{dt} = \frac{-3}{5} \cdot \frac{16}{25}$

$= \frac{-48}{125}$ rad/sec

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7. (18 points) Let $f(x) = (x+7)^{2/3} - 4$.

(a) What is the domain of $f(x)$? Domain $= \mathbb{R}$

$f = (x+7)^{2/3} - 4 = \sqrt[3]{(x+7)^2} - 4$

(b) Determine the end behavior of $f(x)$ and find all horizontal and vertical asymptotes.

$$\left.\begin{aligned} \lim_{x\to\infty} (x+7)^{2/3} - 4 &= \infty \\ \lim_{x\to-\infty} (x+7)^{2/3} - 4 &= \infty \end{aligned}\right\}$$ there are No asymptotes!

(c) Compute $f'(x)$.

$$f'(x) = \frac{2}{3}(x+7)^{-1/3} = \frac{2}{3\sqrt[3]{x+7}}$$

(d) Compute $f''(x)$.

$$f''(x) = -\frac{2}{9}(x+7)^{-4/3} = \frac{-2}{9\sqrt[3]{(x+7)^4}}$$

(e) Find all critical points of $f(x)$ and classify each as a local minimum, local maximum, or neither.

$f' = 0$ N/A

$f' = \emptyset$ $x = -7$

sign of f': $-$ | $+$

$\searrow$ -7 $\nearrow$

local min

(f) Determine the intervals where $f(x)$ is increasing and the intervals where it is decreasing. Write your final answer in the spaces provided below.

increasing $(-7, \infty)$, decreasing $(-\infty, -7)$

Increasing: $(-7, \infty)$ Decreasing: $(-\infty, -7)$

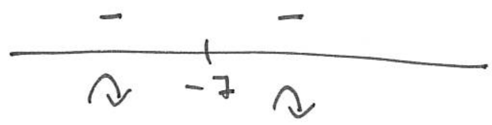
7. (continued.)

(g) Find all inflection points of $f(x)$.

$f'' = 0$ N/A

$f'' = \varphi$ $x = -7$

sign f''



There are no inflection points.

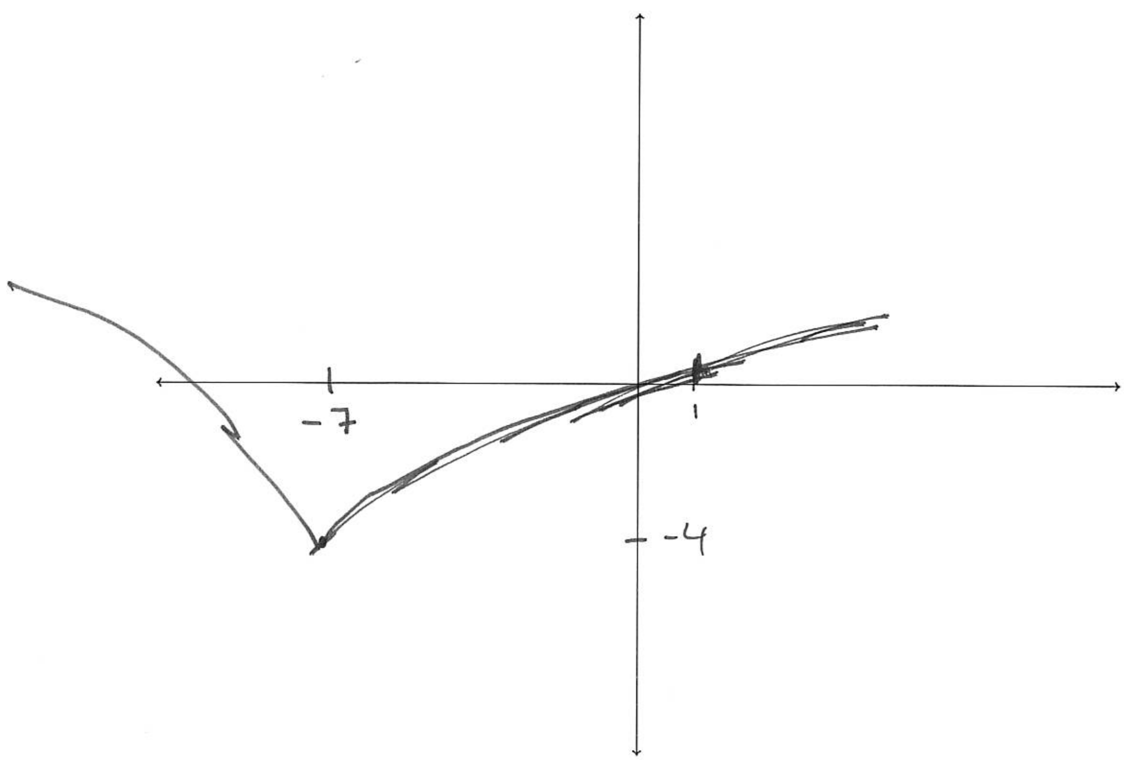
(h) Determine the intervals where $f(x)$ is concave up and the intervals where it is concave down. Write your final answer in the spaces provided below.

concave down on $(-\infty,-7) \cup (-7,\infty)$
or $(-\infty,\infty)$

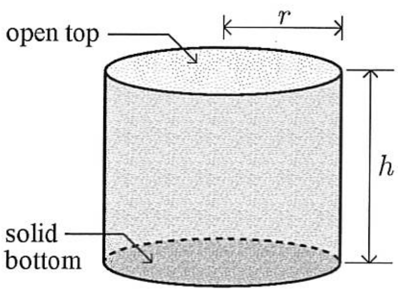
Conc. up: —

Conc. down: $(-\infty,\infty)$ or $(-\infty,-7) \cup (-7,\infty)$

(i) On the axes below, sketch a graph of $f(x)$ that is consistent with all of the features identified above. Label the x and y coordinates of each local extreme point, each inflection point, and of the x-intercepts (if any).



8. (12 points) The sides and bottom of a cylindrical cup of radius r and height h are to be made from a sheet of aluminum. See the diagram below. The total area of material to be used is A. What dimensions for the cup maximize its interior volume? Give a formula for h and for r, each in terms of A.



Area and volume formulas:

- $2\pi rh$ = lateral area of cylinder of radius r, height h
- πr^2 = area of circle of radius r
- $\pi r^2 h$ = volume of cylinder of radius r, height h

$V = \pi r^2 h$ subject to $A = 2\pi rh + \pi r^2$

$\Rightarrow h = \dfrac{A - \pi r^2}{2\pi r}$

$f(r) = \text{volume} = \cancel{\pi} r^{\cancel{2}} \cdot \left(\dfrac{A - \pi r^2}{2\cancel{\pi}\cancel{r}} \right)$

to find the domain, $h \geq 0$,
when $h = 0$ $A = \pi r^2$
or $r = \sqrt{\frac{A}{\pi}}$

$\Rightarrow$ Domain $= \left(0, \sqrt{\frac{A}{\pi}}\right)$

$f(r) = \frac{1}{2} A \cdot r - \frac{\pi}{2} r^3$

$f'(r) = \frac{1}{2} A - \frac{3}{2} \pi r^2$

$f' = 0 \Rightarrow \frac{1}{2} A = \frac{3}{2} \pi r^2 \Rightarrow r^2 = \frac{A}{3\pi} \Rightarrow r = \sqrt{\frac{A}{3\pi}}$

sign of f': $+$ before $\sqrt{\frac{A}{3\pi}}$, $-$ after

global max @ $\boxed{r = \sqrt{\frac{A}{3\pi}}}$

$$h = \frac{A - 2\pi\left(\sqrt{\frac{A}{3\pi}}\right)^2}{2\pi r}$$

$$= \frac{A - \frac{2\cancel{\pi}}{3\cancel{\pi}} A}{2\pi r} = \frac{\frac{1}{3} A}{2\pi r} = \boxed{\frac{A}{6\pi r} = h}$$

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