Math 180 **Name (Print):** ____________________
Spring 2015
Second Midterm
3/11/2015
Time Limit: 2 Hours

This exam contains 11 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

The following rules apply:

- You are expected to abide by the University's rules concerning Academic Honesty.
- You may *not* use your books, notes, or any electronic device including cell phones.
- You must show all of your work. An answer, right or wrong, without the proper justification will receive little to no credit.
- You must complete your work in the space provided. No extra paper will be provided.

Circle your instructor.

- Cabrera
- Cohen
- Groves
- Kobotis
- Lowman
- Shulman
- Steenbergen

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	15	
8	10	
9	15	
Total:	100	

TA Name: ____________________

1. (10 points) An olympic diver enters the water causing a circular ripple to form whose radius increases at 6 feet per second. How fast is the area, enclosed by the ripple, increasing 2 seconds after the diver enters the water?

SOLUTION: After 2 seconds, the radius is $r = 2(6) = 12$ feet. We have $A = \pi r^2$ and $\dfrac{dr}{dt} = 6$. Then

$$\begin{aligned} \frac{dA}{dt} &= 2\pi r \cdot \frac{dr}{dt} \\ \frac{dA}{dt} &= 2\pi(12)(6) = 144\pi. \end{aligned}$$

The ripple's area is increasing at 144π square feet per second.

GRADING RUBRIC:

The following parts are graded independently of one another:

1 point – If the student either labelled or used $\dfrac{dr}{dt} = 6$

2 points – If the student determined that the radius was 12 feet

4 points – If the student had the equation $A = \pi r^2$ AND differentiated with respect to time t

2 point – If the student evaluated $\dfrac{dA}{dt}$

1 point – If the student had proper units on the answer

2. (10 points) Differentiate the following function. Do not simplify your answer. $\tan^{-1}\left(\sqrt{x^2-1}\right)$

SOLUTION: Using the chain rule (twice)

$$\frac{d}{dx}\left(\tan^{-1}\left(\sqrt{x^2-1}\right)\right) = \frac{1}{1+\left(\sqrt{x^2-1}\right)^2} \cdot \frac{1}{2}(x^2-1)^{-1/2} \cdot 2x$$

GRADING RUBRIC:

If no attempt was made to use the chain rule, then the student received 0 points.

If the student made an attempt at using the chain rule (note that $f'(g'(x))$ does not count as an attempt at the chain rule):

5 points – If the student had $\dfrac{1}{1+(\sqrt{x^2-1})^2}$

3 points – If the student had $\dfrac{1}{2}(x^2-1)^{-1/2}$

2 points – If the student had $2x$

-2 points – For each simple error or algebraic mistake the student made

3. (10 points) If $y = x^{-x}$, find $\frac{dy}{dx}$.

SOLUTION: Taking the natural logarithm, we have $\ln y = -x \ln x$. Using implicit differentiation

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= -\ln x + (-x) \cdot \frac{1}{x} \\ \frac{dy}{dx} &= y\,(-\ln x - 1) \\ \frac{dy}{dx} &= x^{-x}\,(-\ln x - 1) \end{aligned}$$

GRADING RUBRIC:

3 points – If the student rewrites the equation as $\ln y = -x \ln x$

6 points – If the student calculates the derivative of $\ln y$ or $-x \ln x$ BUT NOT BOTH

9 points – If the student calculates the derivative of BOTH $\ln y$ and $-x \ln x$

10 points – If the student rewrites the the final answer in terms of only x

OR

2 points – If the student rewrites the equation as $y = e^{\ln(x^{-x})}$

5 points – If the student rewrites the equation as $y = e^{-x \ln x}$.

7 points – If the student calculates the derivative of $e^{-x \ln x}$ but has some errors

10 points – If the student calculates the derivative of $e^{-x \ln x}$

4. (10 points) Find the critical points of $f(x) = x^{2/3}e^x$.

SOLUTION: The domain of f is $(-\infty, \infty)$. Then $f'(x) = \frac{2}{3}x^{-1/3}e^x + x^{2/3}e^x$. Notice that $f'(0)$ does not exist, so $x = 0$ is a critical point. The remaining critical points come from where $f'(x) = \frac{2}{3}x^{-1/3}e^x + x^{2/3}e^x = 0$. Multiplying through by $x^{1/3}$, we have

$$\begin{aligned} \frac{2}{3}e^x + xe^x &= 0 \\ e^x\left(\frac{2}{3} + x\right) &= 0 \\ x &= -\frac{2}{3}. \end{aligned}$$

Therefore the two critical points are $x = 0$ and $x = -\frac{2}{3}$.

GRADING RUBRIC:

4 points – If the student found $f'(x)$

ONLY IF THE STUDENT GOT $f'(x)$ CORRECT CAN S/HE EARN MORE POINTS

7 points – If the student found one of the two critical points but not both

10 points – If the student found both critical points

-2 points – If the student makes a small algebra error

Note: The student did not receive credit for saying $x = 0$ is a critical point if they did not also say that it is because $f'(0)$ is undefined.

5. (10 points) Find the absolute maximum and absolute minimum of $g(x) = \dfrac{x}{x^2+1}$ on $[0, 3]$.

SOLUTION: Since g is continuous on the closed interval $[0, 3]$, the absolute extrema occur at either the endpoints of the interval or a critical point inside the interval.

$$g'(x) = \frac{(x^2+1)\cdot 1 - x\cdot(2x)}{(x^2+1)^2} = 0$$

$$\begin{aligned} x^2+1-2x^2 &= 0 \\ 1-x^2 &= 0 \\ x &= \pm 1. \end{aligned}$$

Since $x = -1$ is not in the interval, we exclude it. Therefore the absolute extrema occur at either $x = 0$, $x = 3$, or $x = 1$:

$$\begin{aligned} g(0) &= 0 \\ g(3) &= \frac{3}{10} \\ g(1) &= \frac{1}{2} \end{aligned}$$

The absolute minimum is 0 and the absolute maximum is $\frac{1}{2}$.

GRADING RUBRIC:

3 points – If the student found $f'(x)$

6 points – If the student set $f'(x) = 0$ and found the critical point

THE REMAINING POINTS CAN ONLY BE GIVEN IF THE STUDENT HAS FOUND THE CORRECT CRITICAL POINT

8 points – If the student evaluated the critical point and endpoints in $g(x)$

10 points – If the student made the correct conclusion about absolute extrema

6. (10 points) If $f(x) = \cos x + 3x$, find $(f^{-1})'(1)$.

SOLUTION: Since $(0, 1)$ is a point on $y = f(x)$, we know $(1, 0)$ is a point $y = f^{-1}(x)$. Then $f'(x) = -\sin x + 3$, so

$$\left(f^{-1}\right)'(1) = \frac{1}{f'(0)} = \frac{1}{3}.$$

GRADING RUBRIC:

0 points – If the student uses the incorrect formula $(f^{-1})'(1) = \dfrac{1}{f'(1)}$

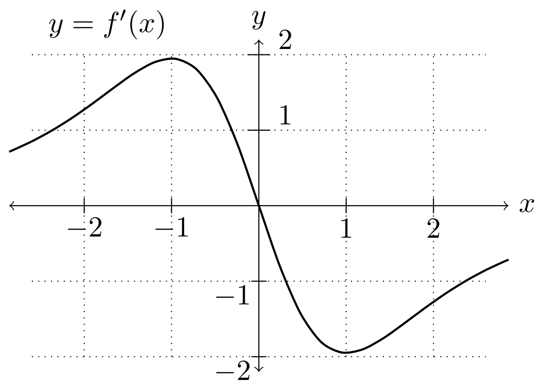
2 points – If the student attempted to use the correct formula $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

4 points – If the students says that $(0, 1)$ is a point on $y = f(x)$ or that $(1, 0)$ is a point on $y = f^{-1}(x)$

8 points – If the student utilizes the formula $(f^{-1})'(1) = \dfrac{1}{f'(0)}$ but has a small error

10 points – If the student utilizes the formula $(f^{-1})'(1) = \dfrac{1}{f'(0)}$ and has the correct derivative

7. (15 points) The picture below is the **graph of the derivative** of $f(x)$. Answer the questions below the graph.



(a) State the interval(s) where $f(x)$ is increasing. Give a one sentence explanation for your answer.

(b) State the interval(s) where $f(x)$ is concave down. Give a one sentence explanation for your answer.

(c) State the point(s) where $f(x)$ has a local maximum or local minimum and what type of local extrema each is. Give a one sentence explanation for your answer.

Solution: (a) $f(x)$ is increasing on $(-\infty, 0)$ since $f'(x) > 0$ on this interval.

(b) $f(x)$ is concave down on $(-1, 1)$ since this is where $f'(x)$ is decreasing.

(c) The only critical point is $x = 0$ since this is where $f'(x) = 0$ and it is a local maximum since $f'(x)$ changes from positive to negative.

Grading Rubric:

For parts (a) and (b), the following scale was used:

2 points – If the correct interval was stated

5 points – If the correct interval was stated with proper justification

For part (c), the following scale was used:

2 points – If $x = 0$ was stated as a critical point

3 points – If $x = 0$ was stated as a critical point and classified

5 points – If $x = 0$ was stated as a critical point and classified with proper justification

NOTE: An answer involving an interval such as $(3, 0)$ or $(2, 0)$ was considered the same as the interval $(-\infty, 0)$. Any values of the function outside of $[-2, 2]$ were not considered (either right or wrong).

8. (10 points) Find the slope of the tangent line to $x^2 + xy - y^2 = -5$ at the point $(-1, 2)$.

SOLUTION: Using implicit differentiation,

$$\begin{aligned} 2x + y + x \cdot \frac{dy}{dx} - 2y \cdot \frac{dy}{dx} &= 0 \\ 2(-1) + 2 + (-1)\frac{dy}{dx} - 2(2)\frac{dy}{dx} &= 0 \\ -5\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= 0 \end{aligned}$$

GRADING RUBRIC:

This problem will be graded in two parts, independent of one another.

The first part, worth 6 points, is the derivative of the equation with respect to x.

1 point – If the student found the derivative of x^2

2 points – If the student found the derivative of xy

2 points – If the student found the derivative of y^2

1 point – If the student found the derivative of 5

The second part, worth 4 points, is finding $\frac{dy}{dx}$ at $(-1, 2)$.

2 points – If the student plugged in $(-1, 2)$ and solved for $\frac{dy}{dx}$ but had an arithmetic error

4 points – If the student plugged in $(-1, 2)$ and solved for $\frac{dy}{dx}$

NOTE: If the first part of the solution was incorrect, the student could earn at most 2 points on the second part

9. (15 points) For each statement below, in the space provided, explain what the statement means in terms of the graph of $f(x)$ *in one sentence.*

SOLUTION:

(a) The domain of f is $[0, \infty)$
Explanation: The graph only exists in Quadrants 1 and/or 4.

(b) f is continuous on its domain
Explanation: The graph can be drawn from left to right without lifting one's pencil.

(c) $f'(2)$ does not exist
Explanation: There is a cusp or corner at $x = 2$.

(d) $f(2) = 0$
Explanation: The graph goes through the point $(2, 0)$.

(e) $f'(x) < 0$ on $(0, 2)$
Explanation: f is decreasing on $(0, 2)$.

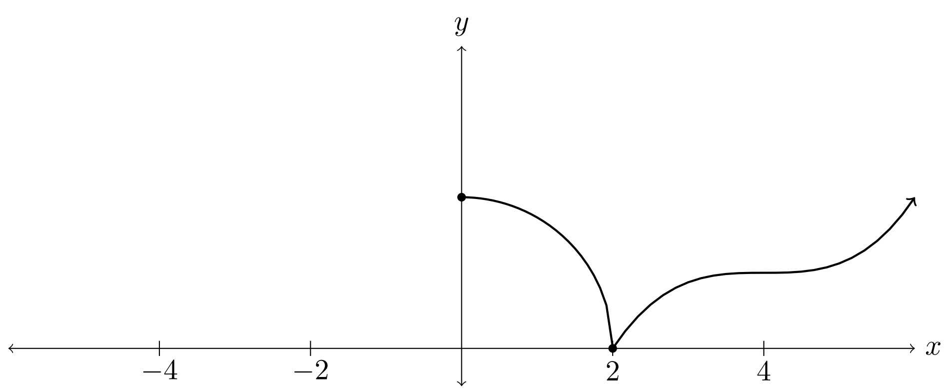
(f) $f'(x) > 0$ on $(2, 4)$ and $f'(x) > 0$ on $(4, \infty)$
Explanation: f is increasing on $(2, 4)$ and on $(4, \infty)$.

(g) $f'(4) = 0$
Explanation: There is a horizontal tangent line at $x = 4$.

(h) $f''(x) < 0$ on $(0, 2)$
Explanation: f is concave down on $(0, 2)$.

On the axes provided, sketch a graph of $f(x)$ that satisfies all of the properties above.

Here is one sketch.



GRADING RUBRIC:

Each explanation is worth 1 point.

1 point – If the student has the proper explanation

The following scale was used to grade the graph, out of 7 points:

2 points – If the graph depicts 2 or 3 of properties (c) through (h)

4 points – If the graph depicts 4 or 5 of properties (c) through (h)

7 points – If the graph depicts all of the properties (c) through (h)

ON TOP OF THAT, PROPERTIES (a) and (b) WERE THEN GRADED AS:

−2 points – If the domain of the graph is not $[0, \infty)$

−2 points – If the graph is not continuous

−2 points – If the graph is not a function

Note: The student can not receive lower than a 0 on this part of the problem with the deductions