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MATH 181 Final Exam
December 13, 2018

Directions. Fill in each of the boxes below. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR INSTRUCTOR. Good luck!

- Circle your instructor's name: Simpson Diep Shulman
- VERY IMPORTANT!!! CHECK THAT THE NUMBER AT THE TOP OF EACH PAGE OF YOUR EXAM IS THE SAME. IT IS THE NUMBER PRECEDED BY A POUND (#) SIGN. IF THEY ARE NOT ALL THE SAME, NOTIFY YOUR INSTRUCTOR OR TA RIGHT AWAY.
- All of your work must fit within the boxes on each page for each question. Nothing outside of the box will be graded! If you write outside of the box, there is a good chance that your exam will not be read and therefore not graded.
- You must show all of your work.
- A solution for one problem may not go on another page.

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1. (10 points) Calculate the integral $\int x^2 \sin(x^3 + 2)\, dx$.

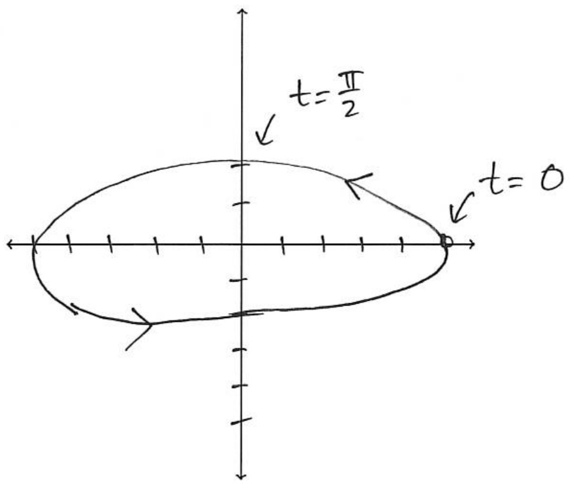
Let $u = x^3 + 2$

$du = 3x^2 dx$

$\int \frac{1}{3} \sin u \, du = -\frac{1}{3} \cos u = -\frac{1}{3} \cos(x^3+2) + C$

2. (10 points) Consider the parametric equations $x = 5\cos t$ and $y = 2\sin t$ for $0 \le t \le 2\pi$.

(a) Sketch a graph of the curve, indicating the orientation.



$\cos t = \frac{x}{5}$

$\sin t = \frac{y}{2}$

$\sin^2 t + \cos^2 t = 1$

$\frac{y^2}{4} + \frac{x^2}{25} = 1$

Ellipse

$t = 0$
$\to (5, 0)$

$t = \frac{\pi}{2}$
$\to (0, 2)$

(b) Find the slope of the tangent line at $t = \frac{\pi}{6}$.

Slope $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-5\sin t} \Rightarrow \frac{2\cos(\pi/6)}{-5\sin(\pi/6)} = \frac{2 \cdot \frac{\sqrt{3}}{2}}{-5 \cdot \frac{1}{2}} = \boxed{-\frac{2\sqrt{3}}{5}}$

3. (20 points) For each of the series below, determine if it converges or diverges and justify your conclusion.

(a) $\sum_{k=1}^{\infty} \frac{k^2}{3k^2+k+5}$

$\lim_{K\to\infty} \frac{K^2}{3K^2+K+5} = \frac{1}{3} \neq 0$ Diverges By Divergence Test

(b) $\sum_{k=1}^{\infty} \frac{6}{3^k} = \sum_{K=1}^{\infty} \left(\frac{1}{3}\right)^K \cdot 6$

$-1 < r = \frac{1}{3} < 1 \Rightarrow$ series converges by Geometric Series Test

(c) $\sum_{k=1}^{\infty} \frac{1}{k!}$ Ratio Test

$\lim_{K\to\infty} \frac{a_{K+1}}{a_K} = \lim_{K\to\infty} \frac{1}{(K+1)!} \cdot \frac{K!}{1} = \lim_{K\to\infty} \frac{1}{K+1} = 0 < 1$

converges by Ratio Test

(d) $\sum_{k=1}^{\infty} \frac{(k-1)!}{k!} = \sum_{K=1}^{\infty} \frac{(K-1)!}{(K-1)!\,K} = \sum_{K=1}^{\infty} \frac{1}{K}$ diverges by Harmonic Series

(e) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}} = \sum_{K=1}^{\infty} \frac{1}{K^{2/3}}$

$p = \frac{2}{3} < 1$ diverges by p-test

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4. (12 points) Find the following indefinite integral $\int \frac{1}{x^2+4x-5}\,dx$.

$$\frac{1}{x^2+4x-5} = \frac{1}{(x+5)(x-1)} = \frac{A}{x+5} + \frac{B}{x-1}$$

$$1 = A(x-1) + B(x+5)$$

$$x=1: \quad 1 = B\cdot 6 \Rightarrow B = \frac{1}{6}$$

$$x=-5: \quad 1 = A(-6) \Rightarrow A = -\frac{1}{6}$$

$$\Rightarrow \int\left(\frac{-1/6}{x+5} + \frac{1/6}{x-1}\right)dx = -\frac{1}{6}\ln|x+5| + \frac{1}{6}\ln|x-1| + C$$

5. (10 points) (a) Find the Maclaurin series for $f(x) = x\cos x$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$x\cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \dots$$

(b) Find the first three non-zero terms of the Maclaurin series for $\cos x - x\sin x$. [Hint: What is the derivative of $f(x)$?]

$$(x\cos x)' = \cos x - x\sin x$$

$$\Rightarrow \cos x - x\sin x = \left[x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \dots\right]'$$

$$= 1 - \frac{3x^2}{2!} + \frac{5x^4}{4!} - \frac{7x^6}{6!} + \dots$$

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6. (15 points) Consider the following linear system

$$\begin{cases} 3x + 5y = 10 \\ -2x + y = 2 \end{cases}$$

(a) Write the system above as an augmented matrix.

$$\left[\begin{array}{cc|c} 3 & 5 & 10 \\ -2 & 1 & 2 \end{array}\right]$$

(b) Perform elementary row operations to solve the system. Make sure you write down each of your elementary operations.

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left[\begin{array}{cc|c} 3 & 5 & 10 \\ -2 & 1 & 2 \end{array}\right] \rightarrow \begin{array}{c} \\ 2R_1 + 3R_2 \end{array} \left[\begin{array}{cc|c} 3 & 5 & 10 \\ 0 & 13 & 26 \end{array}\right] \rightarrow \begin{array}{c} \\ \frac{1}{13}R_2 \end{array} \left[\begin{array}{cc|c} 3 & 5 & 10 \\ 0 & 1 & 2 \end{array}\right]$$

$$\Rightarrow \begin{cases} 3x + 5y = 10 \\ y = 2 \end{cases} \Rightarrow \begin{array}{l} 3x + 5(2) = 10 \Rightarrow x = 0 \\ y = 2 \end{array}$$

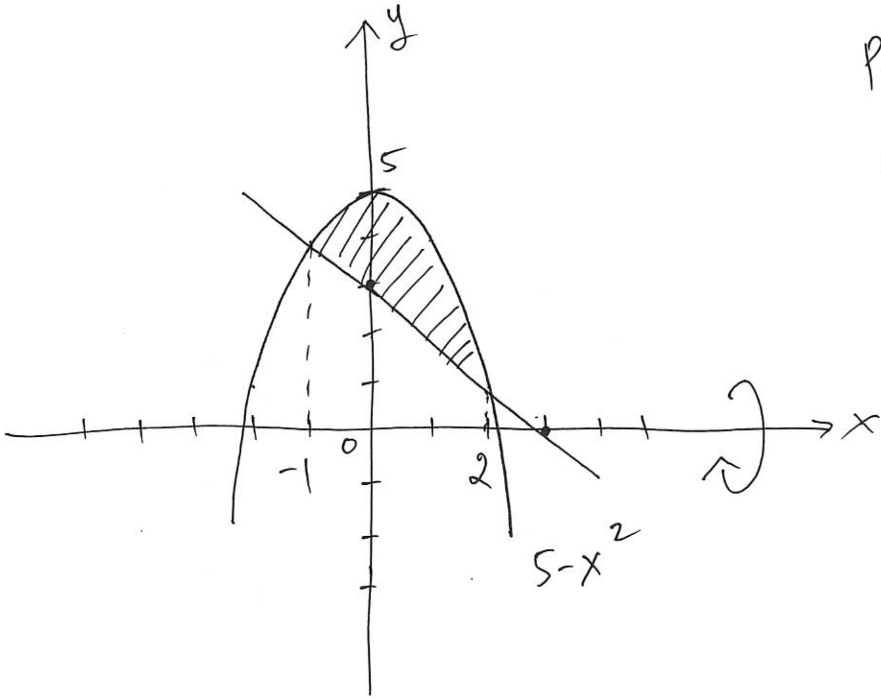
(c) Write the 2×2 coefficient matrix for this system and call it A. Find $\det(A)$ and A^{-1}.

$$A = \begin{bmatrix} 3 & 5 \\ -2 & 1 \end{bmatrix}$$

$$\det(A) = 3(1) - (-2)5 = 3 + 10 = 13$$

$$A^{-1} = \frac{1}{13}\begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1/13 & -5/13 \\ 2/13 & 3/13 \end{bmatrix}$$

7. (12 points) The region bounded by the function $y = 5 - x^2$ and the line $y = -x + 3$ is rotated about the x-axis to form a solid. Find the volume of the resulting solid.



Points of intersection

$$5 - x^2 = -x + 3$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2$$

$$x = -1$$

$$V = \pi \int_{-1}^{2} \left[(5-x^2)^2 - (-x+3)^2 \right] dx$$

$$= \pi \int_{-1}^{2} \left(25 - 10x^2 + x^4 - x^2 + 6x - 9 \right) dx = \pi \int_{-1}^{2} \left(x^4 - 11x^2 + 6x + 16 \right) dx$$

$$= \pi \left(\frac{x^5}{5} - \frac{11x^3}{3} + \frac{6x^2}{2} + 16x \right) \Big|_{-1}^{2}$$

$$= \pi \left[\left(\frac{32}{5} - \frac{88}{3} + 3(4) + 32 \right) - \left(\frac{-1}{5} + \frac{11}{3} + 3 - 16 \right) \right]$$

$$= \pi \left[\frac{33}{5} - \frac{99}{3} + 44 + 13 \right] = \pi \left[\frac{99 - 495 + 855}{15} \right] = \boxed{\frac{459}{15}\pi}$$

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8. (10 points)

(a) Find an antiderivative of $\frac{\ln x}{x^2}$.

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$\int \frac{\ln x}{x^2}\,dx$

$u = \ln x \qquad dv = \frac{1}{x^2}\,dx$

$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{x}$

$uv - \int v\,du$

$\Rightarrow \quad -\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx$

$-\frac{1}{x}\ln x - \frac{1}{x} + C$

(b) Determine if the improper integral $\int_1^{\infty} \frac{\ln x}{x^2}\,dx$ converges or diverges. If it converges, determine it's value.

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{a\to\infty} \int_1^a \frac{\ln x}{x^2}\,dx = \lim_{a\to\infty}\left(-\frac{1}{x}\ln x - \frac{1}{x}\Big|_1^a\right)$$

$$= \lim_{a\to\infty}\left[\left(\underset{\substack{\downarrow\\0}}{\frac{-1}{a}\ln a} - \underset{\substack{\downarrow\\0}}{\frac{1}{a}}\right) - (0-1)\right] = \boxed{1} \text{ converges.}$$

$$\lim_{a\to\infty} \frac{-\ln a}{a} \overset{L'H}{=} \lim_{a\to\infty} \frac{-\frac{1}{a}}{1} = \lim_{a\to\infty} -\frac{1}{a} = 0$$

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9. (15 points) Consider the following series

$$\sum_{n=0}^{\infty} \frac{1}{2^n \cdot n!}$$

(a) Show that the series converges by applying one of our tests.

Ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2^{n+1}(n+1)!} \cdot \frac{2^n \cdot n!}{1} = \lim_{n\to\infty} \frac{\cancel{2^n} \cdot \cancel{n!}}{\cancel{2^n} \cdot 2 \cdot (n+1)\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{1}{2(n+1)} = 0 < 1$$ converges by Ratio Test

(b) Write the Maclaurin series representing $f(x) = e^x$.

$f'(x) = e^x$ evaluating $x = 0 \Rightarrow f'(0) = 1$

$f''(x) = e^x$ $\quad f''(0) = 1$

$\vdots \qquad \vdots$

$$\Rightarrow f(x) = e^x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(c) Use the answer from part (b) to compute the sum of the series.

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{2}\right)^n = e^{\frac{1}{2}} = \sqrt{e}.$$

Let $x = \frac{1}{2}$, $f\left(\frac{1}{2}\right) = e^{\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{2}\right)^k$

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10. (15 points) Consider the series $\sum_{k=1}^{\infty} \frac{3k}{k^2+1}$.

(a) Write out the first three terms in the *series*.

$$\sum_{K=1}^{\infty} \frac{3K}{K^2+1} = \frac{3}{2} + \frac{6}{5} + \frac{9}{10} + \cdots$$

(b) Write out the first three terms in the *sequence of partial sums*.

$$S_1 = a_1 = \frac{3}{2}$$

$$S_2 = a_1 + a_2 = \frac{3}{2} + \frac{6}{5} = \frac{15+12}{10} = \frac{27}{10}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{27}{10} + \frac{9}{10} = \frac{36}{10} = \frac{18}{5}$$

(c) Determine if the series converges or diverges.

Limit Comparison Test

$\sum \frac{3K}{k^2+1} = \sum a_K$, $\sum b_K = \sum \frac{1}{K}$ diverges by Harmonic series

$$L = \lim_{K \to \infty} \frac{a_K}{b_K} = \lim_{K \to \infty} \frac{3K}{K^2+1} \cdot \frac{K}{1} = \lim_{K \to \infty} \frac{3K^2}{K^2+1} = 3$$

Since $0 < L < \infty$ & $\sum \frac{1}{K}$ diverges

Thus $\sum \frac{3K}{K^2+1}$ also diverges

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11. (9 points) Consider the matrix $A = \begin{bmatrix} 5 & 2 \\ 3 & 0 \end{bmatrix}$. For each vector below, determine if it is an eigenvector for A or not (make sure to justify your work). If it is an eigenvector, find its associated eigenvalue.

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$?

$\begin{bmatrix} 5 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$

can't find λ that works

so $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not eigenvector.

(b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 5 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Yes, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector

$\lambda = 6$ is eigenvalue

(c) $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\begin{bmatrix} 5 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} = - \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Yes, $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an eigenvector

$\lambda = -1$ is eigenvalue

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12. (12 points) Consider the polar curve $r = 1 + \cos(2\theta)$ for $0 \leq \theta \leq 2\pi$.

(a) Fill in the second and third columns in the chart below. The column (x, y) should be the rectangular coordinates of the polar point corresponding to r and θ.

θ	r	(x, y)
0	$1 + \cos(0) = 2$	$x = 2\cos 0 = 2$ $y = 2\sin 0 = 0$
$\pi/4$	$1 + \cos(\pi/2) = 1$	$x = 1\cos \pi/4 = \sqrt{2}/2$ $y = 1 \sin \pi/4 = \sqrt{2}/2$
$\pi/2$	$1 + \cos(\pi) = 0$	$x = 0$ $y = 0$
π	$1 + \cos(2\pi) = 2$	$x = 2\cos\pi = -2$ $y = 2\sin\pi = 0$
$7\pi/6$	$1 + \cos(7\pi/3) = 3/2$	$x = \frac{3}{2}\cos\frac{7\pi}{6} = \frac{3}{2}\left(-\frac{\sqrt{3}}{2}\right) = \frac{-3\sqrt{3}}{4}$ $y = \frac{3}{2}\sin\frac{7\pi}{6} = \frac{3}{2}\left(-\frac{1}{2}\right) = \frac{-3}{4}$

(b) On the axes provided, plot the points from part (a) and draw a sketch of the polar curve. You may need to plot more points than what's listed in the table above.