Math 181 Final Exam May 10, 2018

Solutions

1. $$\int_0^{\pi/4} \sin(2x)\,dx = \left.\frac{-\cos(2x)}{2}\right|_0^{\pi/4} = \frac{-\cos\left(\frac{\pi}{2}\right)}{2} - \frac{-\cos(0)}{2}$$

$$= 0 - \frac{-1}{2} = \boxed{\frac{1}{2}}$$

2. $$\int (z+1)\sqrt{z+7}\,dz \underset{\uparrow}{=} \int (u-6)\cdot\sqrt{u}\,du$$

$u = z+7$

$du = dz$

$$= \int u\sqrt{u} - 6\sqrt{u}\,du = \int u^{3/2} - 6u^{1/2}\,du$$

$$= \frac{u^{5/2}}{5/2} - 6\,\frac{u^{3/2}}{3/2} + C$$

$$= \boxed{\frac{2}{5}u^{5/2} - 4u^{3/2} + C}$$

3. $\int_0^{\pi/4} x \cdot \sin(2x)\,dx$

Use IBP:

$u = x \qquad du = dx$

$dv = \sin(2x)\,dx \qquad v = \frac{-\cos(2x)}{2}$

$= x \cdot \frac{-\cos(2x)}{2}\Big|_0^{\frac{\pi}{4}} - \int_0^{\pi/4} \frac{-\cos(2x)}{2}\,dx$

$= \left[\frac{\pi}{4} \cdot \frac{-\cos(\frac{\pi}{2})}{2} - 0 \cdot \frac{-\cos(0)}{2}\right] + \int_0^{\pi/4} \frac{\cos(2x)}{2}\,dx$

$= 0 + \frac{\sin(2x)}{4}\Big|_0^{\frac{\pi}{4}} = \frac{\sin(\frac{\pi}{2})}{4} - \frac{\sin(0)}{4}$

$= \boxed{\frac{1}{4}}$

4. Start with the Ratio Test on $\sum_{k=0}^{\infty} \frac{x^{3k}}{27^k}$:

$$L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{x^{3(k+1)}}{27^{k+1}} \cdot \frac{27^k}{x^{3k}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{x^{\cancel{3k}+3}}{\cancel{x^{3k}}} \cdot \frac{\cancel{27^k}}{27^{\cancel{k+1}}} \right| = \lim_{k\to\infty} \left| \frac{x^3}{27} \right|$$

$$= \frac{|x|^3}{27}$$

We know the series converges when $L<1$, so set $\frac{|x|^3}{27} < 1 \Rightarrow |x|^3 < 27$

$$\Rightarrow |x| < 3 \Rightarrow -3 < x < 3$$

Now test the endpoints:

$x=-3$: $\sum_{k=0}^{\infty} \frac{(-3)^{3k}}{27^k} = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot \cancel{27^k}}{\cancel{27^k}} = \sum_{k=0}^{\infty} (-1)^k$

diverges by divergence test

$x=3$: $\sum_{k=0}^{\infty} \frac{3^{3k}}{27^k} = \sum_{k=0}^{\infty} \frac{27^k}{27^k} = \sum_{k=0}^{\infty} 1$

diverges by divergence test

$\boxed{\text{Interval of Convergence: } (-3, 3)}$

5. $\int x^2 \cdot \ln(x)\,dx$

Use IBP:

$u = \ln(x) \qquad du = \frac{1}{x}\,dx$

$dv = x^2\,dx \qquad v = \frac{1}{3}x^3$

$= \frac{1}{3}x^3 \cdot \ln(x) - \int \frac{1}{3}x^3 \cdot \frac{1}{x}\,dx$

$= \frac{1}{3}x^3 \cdot \ln(x) - \frac{1}{3}\int x^2\,dx$

$= \boxed{\frac{1}{3}x^3 \cdot \ln(x) - \frac{1}{9}x^3 + C}$

6. $\int \frac{13x+4}{(x+3)(2x-1)}\,dx$

Use Partial Fractions:

set $\frac{13x+4}{(x+3)(2x-1)} = \frac{A}{x+3} + \frac{B}{2x-1}$

$13x+4 = A\cdot(2x-1) + B\cdot(x+3)$

Plug in $x = \frac{1}{2}$: $\frac{13}{2} + 4 = B\cdot\left(\frac{1}{2}+3\right) \Rightarrow B = 3$

Plug in $x = -3$: $-39 + 4 = A\cdot(-7) \Rightarrow A = 5$

Thus, $\int \frac{13x+4}{(x+3)(2x-1)}\,dx = \int \frac{5}{x+3} + \frac{3}{2x-1}\,dx$

$= \boxed{5\ln|x+3| + \frac{3}{2}\ln|2x-1| + C}$

7. We know that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

and thus $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$

Therefore,

$$\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$$

$$= \frac{1}{2}\left(\left(1 + \cancel{x} + \frac{x^2}{2!} + \cancel{\frac{x^3}{3!}} + \ldots\right) + \left(1 - \cancel{x} + \frac{x^2}{2!} - \cancel{\frac{x^3}{3!}} + \ldots\right)\right)$$

$$= \frac{1}{2}\left(2 + 2 \cdot \frac{x^2}{2!} + 2 \cdot \frac{x^4}{4!} + 2 \cdot \frac{x^6}{6!} + \ldots\right)$$

$$= \boxed{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots}$$

8. (a) Use the fact that $\sum_{k=0}^{\infty} a \cdot r^k = \frac{a}{1-r}$:

$$\frac{12}{4+x} = \frac{12}{4 \cdot (1 + \frac{x}{4})} = \frac{3}{1 + \frac{x}{4}}$$

$a = 3$

$r = -\frac{x}{4}$

$$= \frac{3}{1 - (-\frac{x}{4})} = \boxed{\sum_{k=0}^{\infty} 3 \cdot \left(-\frac{x}{4}\right)^k}$$

(b) The series converges when $|r| < 1$

$\Rightarrow \left|-\frac{x}{4}\right| < 1 \Rightarrow |x| < 4$. Thus,

the radius of convergence is $\boxed{4}$.

9. (a) the system $\begin{matrix} 3x - 2y = 5 \\ 4x + 7y = 1 \end{matrix}$ can be rewritten as $\begin{bmatrix} 3 & -2 \\ 4 & 7 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

(b) The inverse of $\begin{bmatrix} 3 & -2 \\ 4 & 7 \end{bmatrix}$ is

$$\frac{1}{3\cdot 7 - (-2)\cdot 4} \begin{bmatrix} 7 & 2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} \frac{7}{29} & \frac{2}{29} \\ -\frac{4}{29} & \frac{3}{29} \end{bmatrix}$$

(c) Solving the system:

$$\begin{bmatrix} 3 & -2 \\ 4 & 7 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} \frac{7}{29} & \frac{2}{29} \\ -\frac{4}{29} & \frac{3}{29} \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ 4 & 7 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}}_{A^{-1}\cdot A \cdot \vec{x} = I \cdot \vec{x} = \vec{x}} = \begin{bmatrix} \frac{7}{29} & \frac{2}{29} \\ -\frac{4}{29} & \frac{3}{29} \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5\cdot\frac{7}{29} + \frac{2}{29} \\ 5\cdot\frac{-4}{29} + \frac{3}{29} \end{bmatrix} = \begin{bmatrix} \frac{37}{29} \\ \frac{-17}{29} \end{bmatrix}$$

Thus, $x = \frac{37}{29}$ and $y = -\frac{17}{29}$

10. (a) To find the eigenvalues of $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$, set $\det(A - \lambda I) = 0$ and solve for all possible values for λ.

$$\det\left(\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow (4-\lambda)\cdot(3-\lambda) - 2\cdot 1 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 10 = 0$$

$$\Rightarrow (\lambda-5)(\lambda-2) = 0 \Rightarrow \boxed{\lambda = 2, 5}$$

So, the eigenvalues are 2 and 5

(b) To find an eigenvector $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ for eigenvalue λ, set $A\vec{v} = \lambda\vec{v}$ and solve.

$\lambda = 2$: $A\vec{v} = 2\vec{v} \Rightarrow \begin{bmatrix} 4a + 2b \\ a + 3b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix} \Rightarrow \begin{matrix} 4a+2b = 2a \\ a+3b = 2b \end{matrix}$

Both equations give $a+b=0$, or $a=-b$.

Pick $b=-1$, then $a=1$ and $\boxed{\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

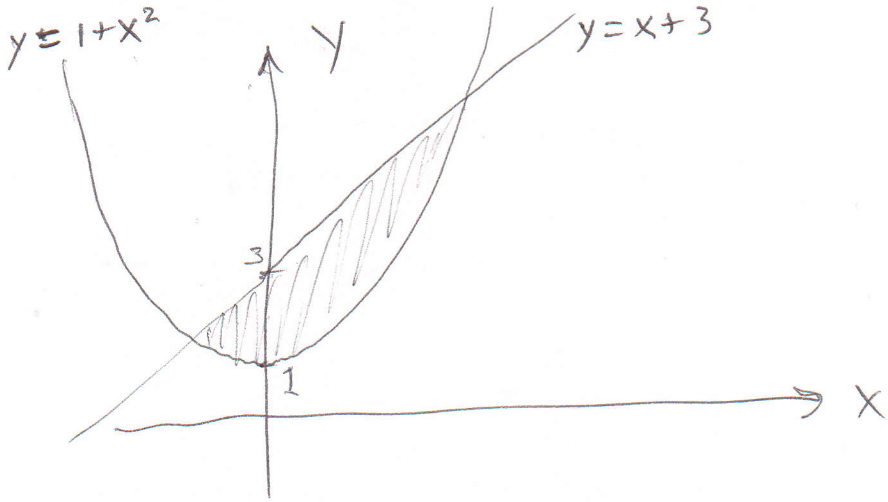
$\lambda = 5$: $A\vec{v} = 5\vec{v} \Rightarrow \begin{bmatrix} 4a + 2b \\ a + 3b \end{bmatrix} = \begin{bmatrix} 5a \\ 5b \end{bmatrix} \Rightarrow \begin{matrix} 4a+2b = 5a \\ a+3b = 5b \end{matrix}$

Both equations give $a-2b=0$, or $a=2b$.

Pick $b=1$, then $a=2$ and $\boxed{\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$

(Note: In both cases, we can pick b to be <u>any</u> nonzero number.)

11. The area bounded by $y = 1 + x^2$ and $y = 3 + x$ is drawn below:



Note that the two curves intersect when $1 + x^2 = 3 + x \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x-2)(x+1) = 0 \Rightarrow x = -1, 2.$

So, the area is given by

$$\int_{-1}^{2} (3+x) - (1+x^2)\, dx$$

$$= \int_{-1}^{2} 2 + x - x^2\, dx$$

$$= \left[2x + \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_{-1}^{2}$$

$$= \left(4 + 2 - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right)$$

$$= \boxed{\frac{9}{2}}$$

12. There is more than one correct answer. One possible answer is

$$x(t) = 3t, \quad y(t) = 2 + 6t, \quad 0 \le t \le 1$$

13. (a) Use $x^2 + y^2 = r^2$ and $\tan\theta = \frac{y}{x}$ to convert.

For $(x, y) = (-3, 3)$, we have

$x^2 + y^2 = (-3)^2 + 3^2 = 18 \Rightarrow r^2 = 18 \Rightarrow r = 3\sqrt{2}$

and $\tan\theta = \frac{3}{-3} = -1 \Rightarrow \theta = \frac{3\pi}{4}$

(since $(-3, 3)$ is in the second quadrant)

For $(x, y) = (-\sqrt{3}, 1)$, we have

$x^2 + y^2 = (-\sqrt{3})^2 + 1^2 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$

and $\tan\theta = -\frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{5\pi}{6}$

(since $(-\sqrt{3}, 1)$ is in the second quadrant)

(b) Use $x = r\cos\theta$, $y = r\sin\theta$ to convert.

For $(r, \theta) = (5, \frac{\pi}{4})$, we have

$x = 5\cos\frac{\pi}{4} \Rightarrow x = \frac{5\sqrt{2}}{2}$ and

$y = 5\sin\frac{\pi}{4} \Rightarrow y = \frac{5\sqrt{2}}{2}$

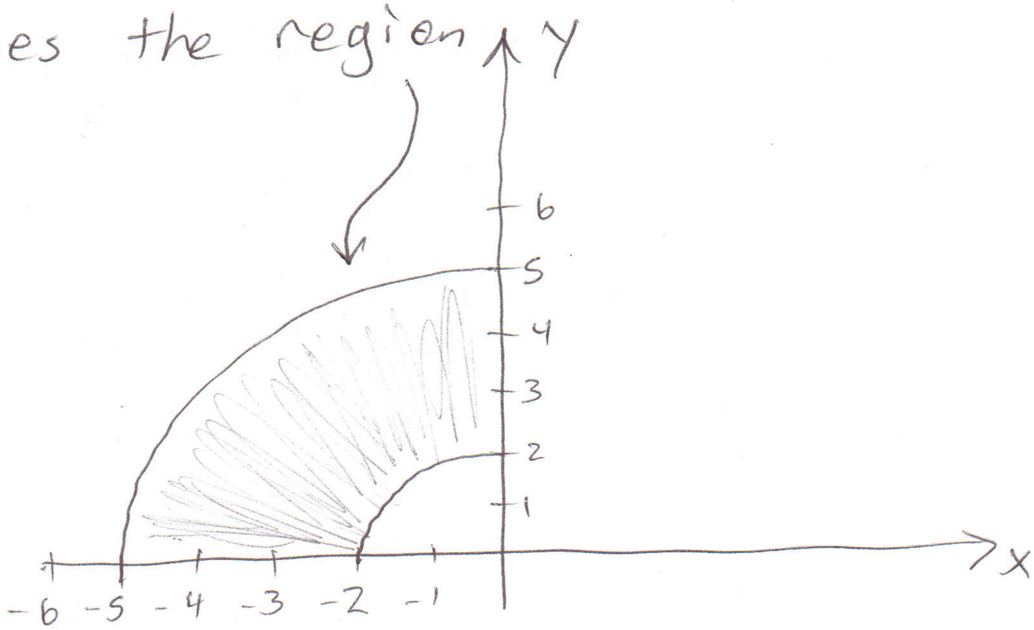
For $(r, \theta) = (-3, -\frac{7\pi}{4})$, we have

$x = -3\cos(-\frac{7\pi}{4}) \Rightarrow x = \frac{-3\sqrt{2}}{2}$ and

$y = -3\sin(-\frac{7\pi}{4}) \Rightarrow y = \frac{-3\sqrt{2}}{2}$

13. (c) $2 \le r \le 5$ and $\frac{\pi}{2} \le \theta \le \pi$

describes the region



14. (a) Note that we can simplify the series as

$$\sum_{k=1}^{\infty} \frac{(-1)^k \cdot 5^k}{2^{3k}} = \sum_{k=1}^{\infty} \frac{(-5)^k}{(2^3)^k} = \sum_{k=1}^{\infty} \left(-\frac{5}{8}\right)^k$$

Thus, the series is geometric with $r = -\frac{5}{8}$. Since $|r| < 1$, the series must converge.

(Other methods we could have used include the Ratio Test and the Alternating Series Test.)

14. (b) First, pull out a minus sign:

$$\sum_{k=0}^{\infty} \frac{k-k^3}{3k^4+k^2+1} = -\sum_{k=0}^{\infty} \frac{k^3-k}{3k^4+k^2+1}$$

Now, all of the terms in the series' are nonnegative, and we can do a limit comparison with the harmonic series $\sum \frac{1}{k}$:

$$L = \lim_{k\to\infty} \frac{\frac{k^3-k}{3k^4+k^2+1}}{\frac{1}{k}}$$

$$= \lim_{k\to\infty} \frac{k^4-k^2}{3k^4+k^2+1} = \frac{1}{3}$$

Since $0 < L < \infty$, and the harmonic series diverges, our series must diverge as well.