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MATH 181 Exam 2
March 15, 2017

Directions. Fill in each of the lines below. Circle your instructor's name and write your TA's name. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR INSTRUCTOR. Good luck!

Print Name: Answers

University Email: ______________________

UIN: ______________________

Circle your instructor's name: Boester Cappetta Steenbergen

TA's Name: ______________________

- VERY IMPORTANT!!! CHECK THAT THE NUMBER AT THE TOP OF EACH PAGE OF YOUR EXAM IS THE SAME. IT IS THE NUMBER PRECEDED BY A POUND (#) SIGN. IF THEY ARE NOT ALL THE SAME, NOTIFY YOUR INSTRUCTOR OR TA RIGHT AWAY.
- All of your work must fit within the boxes on each page for each question. Nothing outside of the box will be graded! If you write outside of the box, there is a good chance that your exam will not be read and therefore not graded.
- A solution for one problem may not go on another page.
- Make clear to the grader what your final answer is.
- Have your student ID ready to be checked when submitting your exam.

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1. (8 points) Given the sequence $a_n = \dfrac{12}{3n-1}$ starting at $n = 1$

(a) Show that the sequence is monotonic.

$$a_{n+1} = \frac{12}{3(n+1)-1} = \frac{12}{3n+2} < \frac{12}{3n-1} = a_n$$

true b/c $3n+2 > 3n-1$ $\Rightarrow \frac{12}{3n+2} < \frac{12}{3n-1}$

$\Rightarrow$ a_n is decreasing!

(b) Find a bound of the sequence.

$$|a_n| = \left|\frac{12}{3n-1}\right| \le 6 = \text{bound}$$

a_1 largest term & $a_1 = \frac{12}{3-1} = 6$

Note $0 \le a_n \le 6$
0 is a lower bound,
6 is an upper bound.

(c) True or False: From parts (a) and (b), the sequence must have a limit.

True, if a_n is monotonic and bounded s.t. $|a_n| \le M$ (M=bound) then the sequence a_n converges.

or here if a_n is decreasing, and a_n has a lower bound, here 0, then a_n must have a limit!

2. (8 points) Find a geometric series with starting term $a = 2$ that converges to a sum of 5. Write the series using $\sum$ notation or give the first four terms of the series. Hint: determine ratio r. DO NOT SIMPLIFY YOUR ANSWER.

a geometric series with $|r| < 1$ converges to

$$\frac{a}{1-r} = \frac{2}{1-r} = 5 \quad \Rightarrow \quad r = \frac{3}{5}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} 2\left(\frac{3}{5}\right)^n = 2 + 2\cdot\frac{3}{5} + 2\cdot\left(\frac{3}{5}\right)^2 + 2\cdot\left(\frac{3}{5}\right)^3 + \ldots$$

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3. (16 points) Determine whether the following series converge or diverge. Justify your answer and name the test(s) used.

(a) $\sum_{k=1}^{\infty} \frac{7k+1}{2k^3-5k-3}$

use limit comparison test to compare to $\sum \frac{1}{k^2}$

note that $\sum \frac{1}{k^2}$ $p=2$, p-test converges

for LCT let $a_k = \frac{7k+1}{2k^3-5k-3}$ and $b_k = \frac{1}{k^2}$, then

$$\lim_{k \to \infty} \frac{a_k}{b_k} = \lim \frac{(7k+1) \cdot k^2}{2k^3-5k-3} = \frac{7}{2} > 0$$

by LCT since $\sum b_k$ converges, so does $\sum a_k$

$\Rightarrow$ the series converges

(b) $\sum_{k=1}^{\infty} \frac{2k^2-7k+9}{5k^2+4k+2}$

$$\lim_{k \to \infty} \frac{2k^2-7k+9}{5k^2+4k+2} = \frac{2}{5} \neq 0$$

by divergence test this series diverges.

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4. (20 points)

(a) Does $\sum_{k=3}^{\infty} \frac{1}{\sqrt[3]{k-2}}$ converge? Justify your answer and name the test(s) used.

use direct comparison: $\boxed{\text{DIVERGES}}$

$$\frac{1}{\sqrt[3]{k-2}} \geq \frac{1}{\sqrt[3]{k}} = \frac{1}{k^{1/3}} \geq 0$$

since $\sum \frac{1}{k^{1/3}}$ diverges, $p=\frac{1}{3}$, p-test, by direct comparison so does $\sum \frac{1}{\sqrt[3]{k-2}}$.

(b) Does $\sum_{k=3}^{\infty} \frac{(-1)^k}{\sqrt[3]{k-2}}$ converge? Justify your answer and name the test(s) used.

$a_k = \frac{1}{\sqrt[3]{k-2}}$ use Alternating Series Test

(i) $a_k \geq 0$ ✓

(ii) $a_{k+1} = \frac{1}{\sqrt[3]{k-1}} \leq a_k = \frac{1}{\sqrt[3]{k-2}}$ decreasing ✓

(iii) $\lim a_k = 0$ ✓

$\Rightarrow$ the series converges

(c) Does $\sum_{k=3}^{\infty} \frac{(-1)^k}{\sqrt[3]{k-2}}$ converge absolutely or conditionally? Briefly explain why.

since the series does not converge absolutely, part(a), and it does converge part (b), it follows that it converges conditionally!

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5. (8 points) Find an expression for the n^{th} partial sum, S_n, of the series

$$\sum_{k=1}^{\infty}\left(\frac{1}{3k-2}-\frac{1}{3k+1}\right)$$

Does this series converge or diverge? Determine its value if it converges.

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{3k-2} - \frac{1}{3k+1}\right)$$

$$= \left(1-\cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{7}}\right) + \left(\cancel{\tfrac{1}{7}} - \cancel{\tfrac{1}{10}}\right) + \cdots + \left(\cancel{\tfrac{1}{3n-2}} - \tfrac{1}{3n+1}\right)$$

$$= 1 - \frac{1}{3n+1}$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{3n+1}\right) = 1$$

$\Rightarrow$ the series converges!

6. (8 points) For some function $f(x)$: $\quad f(3)=1, \quad f'(3)=2, \quad f''(3)=4, \quad f'''(3)=8.$

(a) State the 3rd degree Taylor polynomial of $f(x)$.

$$T_3(x) = 1 + 2(x-3) + 2(x-3)^2 + \frac{4}{3}(x-3)^3$$

(b) Using part (a), approximate $f(3.1)$. DO NOT SIMPLIFY YOUR ANSWER.

$$f(3.1) \approx T_3(3.1) = 1 + 2(0.1) + 2(0.1)^2 + \frac{4}{3}(0.1)^3$$

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7. (18 points) For the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-3)^k}{k5^k}$

(a) Find $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left| \frac{(x-3)^{k+1}}{(k+1)\,5^{k+1}} \cdot \frac{k \cdot 5^k}{(x-3)^k} \right|$

$= \lim_{k\to\infty} \frac{k}{k+1} \cdot \frac{1}{5} \cdot |x-3|$

$= \frac{|x-3|}{5} = L$

(b) Use the result of (a) and the ratio test to find the radius of convergence.

the series converges for $L<1$

i.e. $\frac{|x-3|}{5} < 1 \Rightarrow |x-3| < 5 \Rightarrow -2 < x < 8$

(c) Test the endpoints using the result of (b) to find the interval of convergence.

@ $x=-2$ $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-5)^k}{k \cdot 5^k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-1)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{2k+1}}{k} = \sum_{k=1}^{\infty} \frac{-1}{k}$

or $= -\sum_{k=1}^{\infty} \frac{1}{k}$ which diverges (harmonic series or $p=1$, p-test)

@ $x=8$ $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} 5^k}{k \cdot 5^k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ alternating harmonic series

$\Rightarrow$ converges

$\Rightarrow$ Interval of Convergence $= (-2, 8]$

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8. (14 points) Given the Maclaurin series: $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

(a) State the first 4 terms of the Maclaurin series for $2\tan^{-1}(x^2)$.

$$2\tan^{-1}(x^2) \approx 2x^2 - \frac{2}{3}(x^6) + \frac{2}{5}(x^{10}) - \frac{2}{7}x^{14} + \ldots$$

(b) Using the terms in (a), approximate $\int_0^1 2\tan^{-1}(x^2)\,dx$. DO NOT SIMPLIFY YOUR ANSWER.

$$\approx \int_0^1 2x^2 - \frac{2}{3}x^6 + \frac{2}{5}x^{10} - \frac{2}{7}x^{14}\,dx$$

$$= \frac{2x^3}{3} - \frac{2}{3\cdot7}x^7 + \frac{2}{5\cdot11}x^{11} - \frac{2}{7\cdot15}x^{15} \Big|_0^1$$

$$= \frac{2}{3} - \frac{2}{3\cdot7} + \frac{2}{5\cdot11} - \frac{2}{7\cdot15}$$