MATH 181 Exam 2
March 20, 2019

Directions. Fill in each of the lines below. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR EXAM PROCTOR. This exam contains 8 pages (including this cover page) and 7 problems. After starting the exam, check to see if any pages are missing. Enter all requested information on this page. You are expected to abide by the University's rules concerning Academic Honesty.

TA Name:________________________

The following rules apply:

- You may *not* use your books, notes, calculators, or any electronic device including cell phones. Only pencils/pens allowed.
- You must show all of your work. An answer, right or wrong, without the proper justification will receive little to no credit.
- You *must* complete your work in the space provided. We will be scanning your answers into our grading system, so any work you do that is out of place, too close to the page border, or on the wrong page will *not* be graded!

Circle your instructor.

- Martina Bode
- Robert Cappetta
- Vi Diep
- John Steenbergen

1. (12 points) Clearly circle your answers. If it is not clear which answer you circled, it will be marked incorrect.

(A) (3 points) If $\lim_{k\to\infty} a_k = 0$, then the series $\sum_{k=1}^{\infty} a_k$ converges.

- TRUE
- FALSE

(B) (3 points) When using the Ratio Test on the series $\sum_{k=1}^{\infty} a_k$, if $L = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 0$,

then the series

- Converges
- Diverges
- The ratio test is inconclusive

(C) (3 points) When using the p-Test on the series $\sum_{k=1}^{\infty} \frac{3}{k^3}$, then the series

- Converges
- Diverges
- The p-test is inconclusive

(D) (3 points) Consider the series $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^3+k}$. Which one of the following statements is true? Circle all that are true!

A. *Using the Direct Comparison Test, it is convergent by comparison to* $\sum_{k=1}^{\infty} \frac{1}{k^3}$.

B. *It is divergent by comparing with* $\sum_{k=1}^{\infty} \frac{1}{k}$.

C. *It is divergent because* $\lim_{k\to\infty} \sin^2 k$ *does not exist.*

D. *None of the above.*

2. (24 points) For each series below, determine if it converges or diverges, and state which test you used to form your conclusion. Show all your work!

(a) (8 points) $\sum_{k=1}^{\infty} \frac{2}{3^{k+2}}$ geometric series with $r = \frac{1}{3}$, since $\left|\frac{1}{3}\right| < 1$
the series converges.

(b) (8 points) $\sum_{k=1}^{\infty} \frac{k+1}{2k}$ $\lim \frac{k+1}{2k} = \frac{1}{2} \neq 0$
by divergence test this series diverges

(c) (8 points) $\sum_{k=1}^{\infty} \frac{e^{2k}}{k!}$ ratio test

$$\lim_{k\to\infty} \left| \frac{e^{2k+2}}{(k+1)!} \cdot \frac{k!}{e^{2k}} \right| = \lim_{k\to\infty} \frac{e^2}{k+1} = 0 < 1$$

$\Rightarrow$ the series converges

3. (10 points) Consider $\displaystyle\sum_{k=3}^{\infty}\left(\frac{6}{k-1}-\frac{6}{k+1}\right)$.

(a) (7 points) Find a formula for the nth partial sum $S_n = \displaystyle\sum_{k=3}^{n}\left(\frac{6}{k-1}-\frac{6}{k+1}\right)$ for this series.

$$= \underbrace{\left(\frac{6}{2}-\cancel{\frac{6}{4}}\right)}_{k=3} + \underbrace{\left(\frac{6}{3}-\cancel{\frac{6}{5}}\right)}_{k=4} + \underbrace{\left(\cancel{\frac{6}{4}}-\frac{6}{6}\right)}_{k=5} + \left(\cancel{\frac{6}{5}}-\frac{6}{7}\right)$$

$$+ \cdots + \left(\cancel{\frac{6}{n-2}}-\frac{6}{n}\right) + \left(\cancel{\frac{6}{n-1}}-\frac{6}{n+1}\right)$$

$$= \frac{6}{2}+\frac{6}{3}-\frac{6}{n}-\frac{6}{n+1}$$

$$= 5-\frac{6}{n}-\frac{6}{n+1}$$

(b) (3 points) Using your answer in part (a), determine the value of this series.

$$\rightsquigarrow \sum_{k=3}^{\infty}\left(\frac{6}{k-1}-\frac{6}{k+1}\right) = \lim_{n\to\infty}\left(5-\frac{6}{n}-\frac{6}{n+1}\right) = 5$$

DO NOT WRITE ABOVE THIS LINE

4. (16 points) (a) (7 points) Does the series $\sum_{k=4}^{\infty} \frac{1}{\sqrt{k}-1}$ converge?

Compare to $\sum \frac{1}{\sqrt{k}}$, p-test, $p=\frac{1}{2}$ diverges

$0 < \frac{1}{\sqrt{k}} < \frac{1}{\sqrt{k}-1}$

use direct comparison

thus $\sum \frac{1}{\sqrt{k}-1}$ diverges also

or use Limit Comp Test

$\lim \frac{a_k}{b_k} = \lim \frac{1}{\sqrt{k}-1} \cdot \frac{\sqrt{k}}{1} = 1$

thus by LCT both series diverge

(b) (6 points) Does the series $\sum_{k=4}^{\infty} \frac{(-1)^k}{\sqrt{k}-1}$ converge?

use Alternating Series Test

(i) $b_k = \frac{1}{\sqrt{k}-1} \geq 0$ ✓

(ii) $\lim b_k = 0$ ✓

(iii) $b_{k+1} = \frac{1}{\sqrt{k+1}-1} \leq \frac{1}{\sqrt{k}-1} = b_k$

$\Rightarrow b_k$ decreasing ✓

by AST $\sum \frac{(-1)^k}{\sqrt{k}-1}$ converges

(c) (3 points) Does the series $\sum_{k=4}^{\infty} \frac{(-1)^k}{\sqrt{k}-1}$ converge absolutely, conditionally, or diverge?

since the series does not converge absolutely, but it does converge

$\Rightarrow$ the series converges conditionally

DO NOT WRITE ABOVE THIS LINE

5. (12 points) Consider a function $f(x)$ such that $f(2) = 2$, $f'(2) = -2$, $f''(2) = 4$, and $f'''(2) = 24$

(a) Write the third order Taylor polynomial for $f(x)$ centered at $a = 2$.

$$P_3(x) = 2 - 2(x-2) + \frac{4}{2!}(x-2)^2 + \frac{24}{3!}(x-2)^3$$

$$= 2 - 2(x-2) + 2(x-2)^2 + 4(x-2)^3$$

(b) Use your answer from (a) to estimate $f(2.2)$. You do not need to simplify your answer.

$$f(2.2) \approx 2 - 2(2.2-2) + 2(2.2-2)^2 + 4(2.2-2)^3$$

$$= 2 - 2 \cdot 0.2 + 2 \cdot (0.2)^2 + 4(0.2)^3$$

6. (16 points) Find the interval of convergence of $\sum_{k=1}^{\infty} \frac{(x-3)^k}{(-2)^k k}$. Remember to test the endpoints.

Ratio Test

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(x-3)^{k+1}}{(-2)^{k+1}(k+1)} \cdot \frac{(-2)^k k}{(x-3)^k}\right|$$

$$= \lim_{k\to\infty}\left|\frac{(-2)^k}{(-2)^{k+1}}\right| \cdot \frac{k}{k+1} \cdot \left|\frac{(x-3)^{k+1}}{(x-3)^k}\right|$$

$$= \lim_{k\to\infty} \frac{1}{2} \cdot \frac{k}{k+1} \cdot |x-3|$$

$$= \frac{1}{2}|x-3| < 1$$

$$\Rightarrow |x-3| < 2$$

$$1 < x < 5$$

endpoints:

$x=1$ $\quad \sum \frac{(1-3)^k}{(-2)^k k} = \sum \frac{\cancel{(-2)^k}}{\cancel{(-2)^k} k} = \sum \frac{1}{k}$ diverges

harmonic series

$x=5$ $\quad \sum \frac{(5-3)^k}{(-2)^k k} = \sum \frac{2^k}{(-2)^k k} = \sum \frac{(-1)^k}{k}$ converges

alternating harmonic series

$\Rightarrow$ Interval of Convergence $= (1, 5]$

DO NOT WRITE ABOVE THIS LINE

7. (10 points) Define a function $f(x)$ as the power series $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$. Note that this series converges for all real numbers x.

(a) Find a power series representation of the series of $f(x^2)$.

$$f(x^2) = 1 + (x^2) + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + \cdots$$

$$= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots$$

(b) Use the first three non-zero terms of the series you found in part (a) to write a power series representation for the derivative of $f(x^2)$.

$$(f(x^2))' \approx \left(1 + x^2 + \frac{x^4}{2}\right)'$$

$$= 0 + 2x + \frac{4x^3}{2}$$

$$= 2x + 2x^3$$