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MATH 210 Final Exam

December 10, 2015

Directions. Fill in each of the lines below. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR INSTRUCTOR.

Name: ______________________________

UIN: ______________________________

University Email: ______________________________

Check next to your instructor's name:

Lukina	11am	
Lukina	1pm	
Kobotis	8am	
Greenblatt	10am	
Greenblatt	1pm	
Goldbring	11am	
Hong	10am	
Hong	12pm	
Dumas	12pm	
Dai	2pm	
Heard	9am	
Wang	2pm	
Torres	9am	

- All of your work must fit within the boxes on each page for each question. Nothing outside of the box will be graded!
- A solution for one problem may not go on another page.
- Show all your work. Unjustified answers are not correct. Make clear to the grader what your final answer is.
- Have your student ID ready to be checked when submitting your exam.

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1. **(10pt)** Determine whether the given vectors are orthogonal, parallel, or neither. Justify your answer.

(a) $\mathbf{u} = \langle -1, 5, 2 \rangle$ and $\mathbf{v} = \langle 4, 2, -3 \rangle$.

$\vec{u} \cdot \vec{v} = -4 + 10 - 6 = 0$, orthogonal

(b) $\mathbf{a} = \langle 2, -4 \rangle$ and $\mathbf{b} = \langle -1, 2 \rangle$.

$-2\vec{b} = \langle 2, -4 \rangle = \vec{a}$, parallel

2. **(10pt)** Find an equation of the line which is parallel to the line

$$\mathbf{r}(t) = \langle 1 + t, 2 - t, 3 \rangle$$

and which passes through the point $P(3, 3, 3)$.

$\vec{v} = \langle 1, -1, 0 \rangle$

$\vec{r}(t) = \langle 3 + t, 3 - t, 3 \rangle$

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3. **(10pt)** Consider the function

$$f(x,y) = \frac{x}{y}.$$

(a) Find the unit vector in the direction of the steepest descent at the point $P(6,-2)$.

(b) Find the rate of change in the direction of the steepest descent at the point $P(6,-2)$.

a) $\nabla f = \langle \frac{1}{y}, -\frac{x}{y^2} \rangle$

$-\nabla f(-6,-2) = -\langle -\frac{1}{2}, -\frac{6}{4} \rangle = \langle \frac{1}{2}, \frac{3}{2} \rangle$

the direction of the steepest descend

$\vec{u} = -\frac{\nabla f}{|\nabla f|} = \frac{\langle \frac{1}{2}, \frac{3}{2} \rangle}{\sqrt{\frac{1}{4}+\frac{9}{4}}} = \langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle$

b) $D_{\vec{u}} f(6,-2) = -|\nabla f| = -\frac{\sqrt{10}}{2}$

4. **(10 pt)** Consider the function
$$f(x, y) = xy - 2x - y.$$

(a) Find the critical points of the function.

(b) Use the Second Derivative Test to classify each critical point as a local maximum, local minimum, or a saddle point.

a) $f_x = y - 2 = 0$

$f_y = x - 1 = 0$

$(x, y) = (1, 2)$ is a critical point

b) $f_{xx} = 0, \quad f_{yy} = 0, \quad f_{xy} = 1$

$D(1,2) = 0 - 1^2 = -1 < 0$

saddle point

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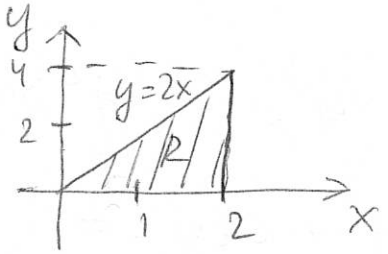
5. **(10 pt)** Evaluate the double integral

$$\iint_R e^{x-y}dA,$$

where R is the triangular region with vertices $(0,0)$, $(2,4)$ and $(2,0)$.



$$\int_0^2 \int_0^{2x} e^{x-y}\,dy\,dx = \int_0^2 -e^{x-y}\Big|_0^{2x} dx =$$

$$\int_0^2 -(e^{x-2x} - e^{x})dx = \int_0^2 (e^{x} - e^{-x})dx =$$

$$(e^{x}+e^{-x})\Big|_0^2 = (e^{2}+e^{-2}) - (1+1) = e^{2}+e^{-2}-2$$

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or: $$\int_0^4 \int_{-\frac{y}{2}}^{2} e^{x-y}\,dx\,dy = \int_0^4 \left(e^{x-y}\Big|_{\frac{y}{2}}^{2}\right)dy =$$

$$\int_0^4 \left(e^{2-y} - e^{\frac{y}{2}-y}\right)dy = \int_0^4 \left(e^{2-y} - e^{-\frac{1}{2}y}\right)dy =$$

$$-e^{2-y}\Big|_0^4 + 2e^{-\frac{1}{2}}\Big|_0^4 = -e^{-2}+e^{2}+2e^{-2}-2 =$$

$$e^{2}+e^{-2}-2$$

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6. **(10 pt)** Consider a conservative vector field

$$\mathbf{F}(x, y) = 2xy^3\ \mathbf{i} + 3x^2y^2\ \mathbf{j}.$$

(a) Find a potential function for $\mathbf{F}$.

(b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a curve from $(2, 1)$ to $(3, 2)$.

$\nabla \varphi = \langle \varphi_x, \varphi_y \rangle = \langle f, g \rangle = \langle 2xy^3, 3x^2y^2 \rangle$

a) $\varphi = \int 2xy^3\,dx = 2\frac{x^2}{2}y^3 + c(y) =$

$x^2y^3 + c(y)$

$\varphi_y = 3x^2y^2 + c'(y) = 3x^2y^2,$

so $c'(y) = 0,$ and $c(y) = k.$

We take $k = 0$, then

$\varphi = x^2y^3.$

b) $\int_C \vec{F} \cdot d\vec{r} = \varphi(3,2) - \varphi(2,1) =$

$3^2 \cdot 2^3 - 2^2 \cdot 1^3 = 9 \cdot 8 - 4 = 72 - 4 = 68$

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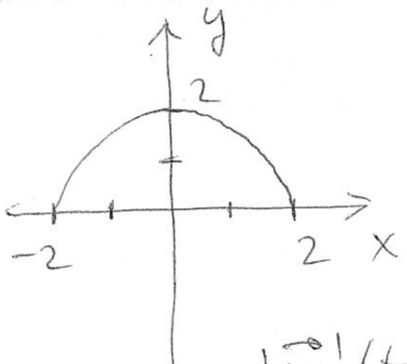
7. **(10 pt)** Evaluate the line integral

$$\int_C (x+y)ds,$$

where C is the half of circle $x^2 + y^2 = 4$ in the upper half plane.



$\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle,$
$0 \le t \le \pi$

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t \rangle$

$|\vec{r}\,'(t)| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$

$\int_C (x+y)ds = \int_0^\pi (2\cos t + 2\sin t) \cdot 2 \cdot dt =$

$(4\sin t - 4\cos t)\Big|_0^\pi =$

$4(\sin\pi - \cos\pi) - 4(\sin 0 - \cos 0) = 4+4 = 8.$

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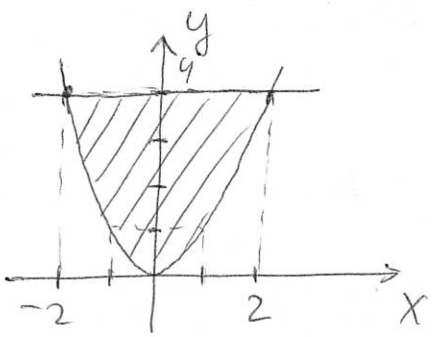
8. **(10 pt)** Use Green's theorem to evaluate the line integral

$$\int_C (y^2 - \tan^{-1} x)dx + (3x + \sin y)dy,$$

where C is the boundary of the region enclosed by the parabola $y = x^2$ and the line $y = 4$, oriented counterclockwise.



$$f = y^2 - \tan^{-1} x$$

$$g = 3x + \sin y$$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 3 - 2y$$

$$\int_C (y^2 - \tan^{-1} x)dx + (3x + \sin y)dy = \iint_D (3-2y)\,dA =$$

$$\int_{-2}^{2}\int_{x^2}^{4} (3-2y)\,dy\,dx = \int_{-2}^{2} (3y - y^2)\Big|_{x^2}^{4} dx =$$

$$\int_{-2}^{2} (-4) - (3x^2 - x^4)\,dx = 2\int_0^2 (x^4 - 3x^2 - 4)\,dx =$$

$$2\left(\frac{x^5}{5} - x^3 - 4x \Big|_0^2\right) = 2\left(\frac{32}{5} - 8 - 8\right) =$$

$$-\frac{96}{5}$$

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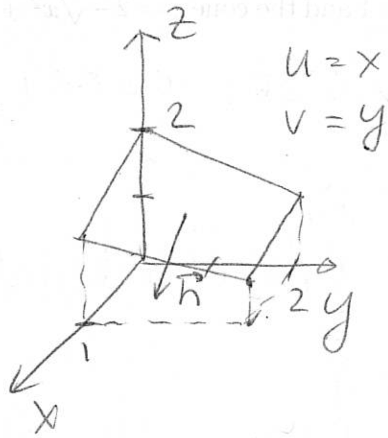
9. **(10 pt)** The slanted roof of a building is given by the equation $z = 2 - \frac{1}{2}x - \frac{1}{2}y$, above the rectangle $0 \le x \le 1$, $0 \le y \le 2$ in the horizontal plane. The flow of the rain is described by the vector field $\mathbf{F} = \langle -1, -2, -1 \rangle$. Find the downward flux of the rain through the roof.



$u = x$

$v = y$

$\vec{r}(u,v) = \langle u, v, 2 - \frac{1}{2}u - \frac{1}{2}v \rangle,$

$0 \le u \le 1, \quad 0 \le v \le 2$

$\vec{t}_u = \langle 1, 0, -\frac{1}{2} \rangle, \quad \vec{t}_v = \langle 0, 1, -\frac{1}{2} \rangle$

$\vec{t}_u \times \vec{t}_v = \langle \frac{1}{2}, \frac{1}{2}, 1 \rangle$

must reverse the orientation

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_R \langle -1, -2, -1 \rangle \cdot \langle -\frac{1}{2}, -\frac{1}{2}, -1 \rangle dA =$$

$$\iint_R \left(\frac{1}{2} + 1 + 1\right) dA = \frac{5}{2} \int_0^1 \int_0^2 dy\, dx =$$

$$\frac{5}{2} \int_0^1 y \Big|_0^2 dx = 5 \int_0^1 dx = 5$$

or

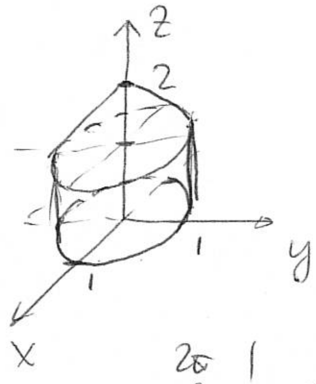
$$\frac{5}{2} \int_0^1 \int_0^2 dy\, dx = \frac{5}{2} \cdot 2 = 5$$

the area of a rectangle

10. **(10 pt)** Evaluate the integral

$$\iiint_D (2 - x^2 - y^2)dV,$$

where D is the solid bounded by the xy-plane, the cylinder $x^2+y^2 = 1$ and the cone $z = 2-\sqrt{x^2+y^2}$.



$D = \{(r,\theta,z) : 0 \le \theta \le 2\pi,\ 0 \le r \le 1,$
$0 \le z \le 2-r\}$

$$\iiint_D (2-x^2-y^2)dV = \int_0^{2\pi}\int_0^1\int_0^{2-r}(2-r^2)r\,dz\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^1 (2r-r^3)z\Big|_0^{2-r} dr\,d\theta = \int_0^{2\pi}\int_0^1 (2r-r^3)(2-r)\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^1 (4r-2r^3-2r^2+r^4)\,dr\,d\theta =$$

$$\int_0^{2\pi}\left(4\frac{r^2}{2} - 2\cdot\frac{r^4}{4} - 2\frac{r^3}{3} + \frac{r^5}{5}\right)\Big|_0^1 d\theta =$$

$$\left(2-\frac{1}{2}-\frac{2}{3}+\frac{1}{5}\right)\theta\Big|_0^{2\pi} = 2\pi\left(\frac{3}{2}-\frac{2}{3}+\frac{1}{5}\right) =$$

$$\frac{31\pi}{5}$$