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MATH 210 Final Exam

December 8, 2016

Directions. Fill in each of the lines below. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR INSTRUCTOR.

Name: ____________________

UIN: ____________________

University Email: ____________________

Check next to your instructor's name:

Lukina	10am	
Lukina	11am	
Steenbergen	11am	
Steenbergen	12pm	
Kobotis	8am	
Sparber	2pm	
Leslie	2pm	
Awanou	3pm	
Heard	9am	
Woolf	9am	
Abramov	12pm	
Sinapova	3pm	
Hong	10am	
Freitag	1pm	
Greenblatt	1pm	

- All of your work must fit within the boxes on each page for each question. Nothing outside of the box will be graded! If you write outside of the box, there is a good chance that your solution will not be read and therefore not graded.
- A solution for one problem may not go on another page.
- Show all your work. Unjustified answers are not correct. Make clear what your final answer is.
- Have your student ID ready to be checked when submitting your exam.

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1. **(10 pt)** For the vectors $\mathbf{u} = \langle 1, 2, -1 \rangle$ and $\mathbf{v} = \langle -2, 4, 4 \rangle$

   (a) Compute the dot product $\mathbf{u} \cdot \mathbf{v}$.

   (b) Compute the cross product $\mathbf{u} \times \mathbf{v}$.

   (c) Determine if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Justify your answer.

   (d) Determine if $\mathbf{u}$ and $\mathbf{v}$ are parallel. Justify your answer.

a) $\vec{u} \cdot \vec{v} = \langle 1, 2, -1 \rangle \cdot \langle -2, 4, 4 \rangle = -2 + 8 - 4 = 2$

b) $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ -2 & 4 & 4 \end{vmatrix} = \vec{i}\,(8 - (-4)) - \vec{j}\,(4 - 2) + \vec{k}\,(4 - (-4)) = \langle 12, -2, 8 \rangle$

c) not orthogonal, since $\vec{u} \cdot \vec{v} \neq 0$.

d) not parallel, since $\vec{u} \times \vec{v} \neq \vec{0}$.

or: not parallel, since there is no c such that $c \langle 1, 2, -1 \rangle = \langle -2, 4, 4 \rangle$ (by inspection).

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2. **(15 pt)** Consider the function

$$f(x, y) = x^2 + 2y^2.$$

(a) Find the critical points of $f(x, y)$.

(b) Find the absolute maximum and the absolute minimum of $f(x, y)$ on the region R, where

$$x^2 + y^2 \leq 4.$$

a) $f_x = 2x = 0$ crit. point $(0,0)$
$f_y = 4y = 0$

b) $\nabla f(x,y) = \langle 2x, 4y \rangle$
$g(x,y) = x^2 + y^2 - 4 = 0$ $\nabla g(x,y) = \langle 2x, 2y \rangle$

Lagrange multipliers:

$2x = \lambda\, 2x \Rightarrow x(1-\lambda) = 0$
$4y = \lambda\, 2y$ $x = 0$ or $\lambda = 1$
$x^2 + y^2 = 4$

$x = 0$: $y^2 = 4$, $y = \pm 2$, points $(0,-2), (0,2)$
$\lambda = 1$: $4y - 2y = 0$, so $y = 0$, $x^2 = 4$, $x = \pm 2$,
points $(-2,0), (2,0)$.

$f(0,0) = 0$ $f(-2,0) = 4$ $f(2,0) = 4$
$f(0,-2) = 2 \cdot 4 = 8$ $f(0,2) = 2 \cdot 4 = 8$

abs. max $f(0,-2) = f(0,2) = 8$
abs. min $f(0,0) = 0$.

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3. **(10 pt)**

(a) Find an equation of the plane tangent to the graph of the function $f(x, y) = x^2 + xy$ at the point $(2, 1)$.

(b) Use a) to estimate $f(2.1, 0.8)$.

a) $\nabla f(x,y) = \langle 2x+y, x \rangle$

$\nabla f(2,1) = \langle 4+1, 2 \rangle = \langle 5, 2 \rangle$, $f(2,1) = 4 + 2 \cdot 1 = 6$

eqn. of the tangent plane:

$z = 5(x-2) + 2(y-1) + 6$

b) $L(x,y) = 5(x-2) + 2(y-1) + 6$

$f(2.1, 0.8) = 5(2.1 - 2) + 2(0.8 - 1) + 6 =$

$5 \cdot 0.1 - 2 0.2 + 6 = 0.5 - 0.4 + 6 = 6.1$

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4. **(10 pt)** For the integral

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{1+x^3}\, dx\, dy$$

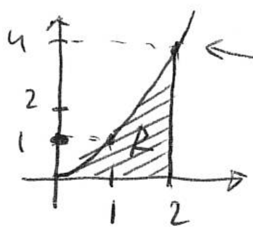
(a) Sketch the region of integration.

(b) Reverse the order of integration.

(c) Evaluate the integral.

$R = \{(x,y) \mid 0 \le y \le 4,\ \sqrt{y} \le x \le 2\}$

a)



$x = \sqrt{y}$

$y = x^2$

b) $R = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le x^2\}$

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{1+x^3}\, dx\, dy = \int_0^2 \int_0^{x^2} \sqrt{1+x^3}\, dy\, dx =$$

$$\int_0^2 \sqrt{1+x^3}\, y \Big|_0^{x^2} dx = \int_0^2 x^2 \sqrt{1+x^3}\, dx =$$

$u = 1 + x^3 \qquad du = 3x^2 dx$

if $x = 0$ then $u = 1$

$x = 2 \qquad u = 1 + 8 = 9$

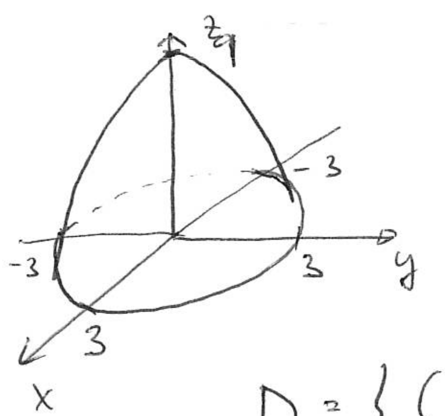
$$\frac{1}{3} \int_1^9 u^{1/2} du = \frac{1}{3} \frac{u^{3/2}}{3/2} \Big|_1^9 = \frac{1}{3} \cdot \frac{2}{3} \left((\sqrt{9})^3 - (\sqrt{1})^3 \right) =$$

$$\frac{2}{9}(27 - 1) = \frac{26 \cdot 2}{9} = \frac{52}{9}$$

5. **(10pt)** Write down an iterated triple integral in cylindrical coordinates, computing

$$\iiint_D (x^2+y^2)^2 dV,$$

where D is the solid, bounded above by the surface $z = 9 - x^2 - y^2$ and below by the plane $z = 0$. Do not evaluate the integral.



the intersection of the surface and the plane:

$0 = 9 - x^2 - y^2$

$x^2 + y^2 = 9$ circle of radius 3

in cylindrical coordinates:

$D = \{ (r, \theta, z) \mid 0 \le \theta \le 2\pi, \ 0 \le r \le 3, \ 0 \le z \le 9 - r^2 \}$

$$\iiint_D (x^2+y^2)^2 dV = \int_0^{2\pi} \int_0^3 \int_0^{9-r^2} r^4 \cdot r \, dz \, dr \, d\theta$$

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6. **(15 pt)** Consider a vector field $\mathbf{F} = \langle ye^{xy}, 1 + xe^{xy} \rangle$.

   (a) Verify that the vector field $\mathbf{F}$ is conservative by checking partial derivatives.

   (b) Find a potential function for $\mathbf{F}$.

a) $f = ye^{xy}$ $\quad g = 1 + xe^{xy}$

$f_y = e^{xy} + yxe^{xy}$ $\quad g_x = e^{xy} + xye^{xy}$

$f_y = g_x$, so $\vec{F}$ is conservative.

b) $\varphi_x = f = ye^{xy}$, $\quad \varphi_y = g = 1 + xe^{xy}$

$\varphi = \int \varphi_x \, dx = \int ye^{xy} \, dx = e^{xy} + c(y)$

$\varphi_y = xe^{xy} + c'(y) = 1 + xe^{xy}$

$c'(y) = 1$

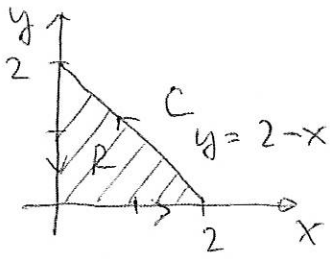
$c(y) = \int dy = y + d$, take $d = 0$

$$\boxed{\varphi = e^{xy} + y}$$

7. **(10pt)** Use Green's theorem to compute the flux of the vector field

$$\mathbf{F}(x,y) = \langle e^{y^3} + x^2, y + e^{x^2+1} \rangle$$

along the triangular curve consisting of the line segments between $(0,0)$ and $(2,0)$, between $(2,0)$ and $(0,2)$, and between $(0,2)$ and $(0,0)$ oriented counterclockwise.



$$\oint_C \vec{F} \cdot \vec{n}\, ds = \iint_R (f_x + g_y)\, dA$$

$f = e^{y^3} + x^2 \qquad f_{\cancel{x}} = 2x$

$g = y + e^{x^2+1} \qquad f_y = 1$

$R = \{ (x,y) \mid 0 \le x \le 2,\ 0 \le y \le 2-x \}$

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \int_0^2 \int_0^{2-x} (2x+1)\, dy\, dx = \int_0^2 (2xy + y)\Big|_0^{2-x} dx =$$

$$\int_0^2 (2x(2-x) + 2 - x)\, dx = \int_0^2 (4x - 2x^2 + 2 - x)\, dx =$$

$$\left(-2\frac{x^3}{3} + 3\frac{x^2}{2} + 2x\right)\Big|_0^2 = -\frac{2}{3}\cdot 8 + \frac{3}{2}\cdot 4 +$$

$$4 - 0 = \frac{-32+36}{6} + 4 = -\frac{4}{6} + 4 = \frac{2}{3} + 4 =$$

$$\frac{10}{3}$$

8. **(10 pt)** Let $\varphi(x, y) = \sqrt{x^2 + y^2 + 1}$.

(a) Find the gradient vector field $\mathbf{F} = \nabla \varphi$.

(b) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ of $\mathbf{F}$ along a curve C starting at $P = (0, 0)$ and ending at $Q = (2, 1)$.

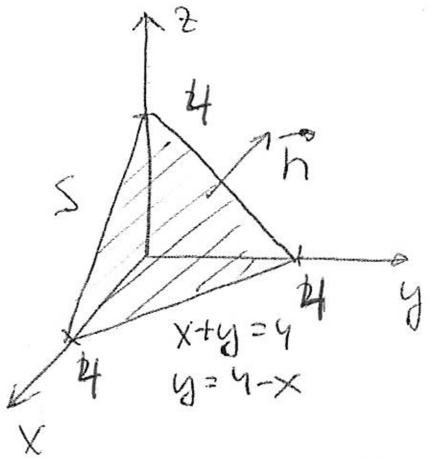
a) $\vec{F} = \nabla \varphi = \left\langle \frac{1}{2} \frac{2x}{\sqrt{x^2+y^2+1}}, \frac{1}{2} \frac{2y}{\sqrt{x^2+y^2+1}} \right\rangle =$

$\left\langle \frac{x}{\sqrt{x^2+y^2+1}}, \frac{y}{\sqrt{x^2+y^2+1}} \right\rangle$

b) $\int_C \vec{F} \cdot d\vec{r} = \varphi(2,1) - \varphi(0,0) =$

$\sqrt{4+1+1} - \sqrt{0+0+1} = \sqrt{6} - 1$

9. **(10 pt)** Let S be the portion of the plane $x + y + z = 4$ in the first octant. Compute the flux of the vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ across S in the upward direction.



$x = u, \quad y = v$

$\vec{r}(u,v) = \langle u, v, 4-u-v \rangle$

$R = \{(u,v) \mid 0 \le u \le 4, \ 0 \le v \le 4-u \}$

$\vec{t}_u = \langle 1, 0, -1 \rangle$

$\vec{t}_v = \langle 0, 1, -1 \rangle$

$$\vec{t}_u \times \vec{t}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i}(1) - \vec{j}(-1) + \vec{k}(1) = \langle 1, 1, 1 \rangle$$

$\vec{F} = \langle u, v, 4-u-v \rangle$

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_R \langle u, v, 4-u-v \rangle \cdot \langle 1, 1, 1 \rangle \, dA =$$

$$\int_0^4 \int_0^{4-u} (u+v+4-u-v)\, dv\, du = \int_0^4 \int_0^{4-u} 4 \, dv\, du =$$

$$\int_0^4 4v \Big|_0^{4-u} du = \int_0^4 (16-4u)\, du = \left(16u - 4\frac{u^2}{2}\right)\Big|_0^4 =$$

$$64 - 2 \cdot 4^2 = 64 - 32 = 12$$