DO NOT WRITE ABOVE THIS LINE!!

MATH 210 Final Exam

May 4, 2017

Directions. Fill in each of the lines below. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR INSTRUCTOR.

Name: ____

UIN: ____

University Email: ____

Check next to your instructor's name:

Lukina		
Abramov		
Heard		
Woolf		
Thulin		
Page		
Skalit		
Kobotis		
Freitag		
Shulman		
Lesieutre		

- All of your work must fit within the boxes on each page for each question. Nothing outside of the box will be graded! If you write outside of the box, there is a good chance that your solution will not be read and therefore not graded.
- A solution for one problem may not go on another page.
- Show all your work. Unjustified answers are not correct. Make clear what your final answer is.
- Have your student ID ready to be checked when submitting your exam.

DO NOT WRITE ABOVE THIS LINE!!

1. **(10 pt)** Find the equation of the plane that contains the points $A(1, 2, -1)$, $B(3, 1, 0)$ and $C(0, 0, 1)$.

$$\vec{AB} = \langle 3-1, 1-2, 0-(-1) \rangle = \langle 2, -1, 1 \rangle$$

$$\vec{AC} = \langle 0-1, 0-2, 1-(-1) \rangle = \langle -1, -2, 2 \rangle$$

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ -1 & -2 & 2 \end{vmatrix} = \vec{i}(-2+2) - \vec{j}(4-(-1)) + \vec{k}(-4-1) = \langle 0, 5, -5 \rangle$$

$$0(x-1) + 5(y-2) - 5(z+1) = 0$$

$$5y - 5z - 10 - 5 = 0$$

$$5y - 5z = 15 \quad \text{or} \quad y - z = 3$$

DO NOT WRITE ABOVE THIS LINE!!

2. **(15 pt)** Consider the function

$$f(x, y) = 2x^2 + y^2.$$

(a) Compute the directional derivative in the direction of the vector $\mathbf{u} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$ at $(2, -1)$.

(b) Find the unit vector in the direction of the steepest ascent and the rate of the steepest ascent at $(2, -1)$.

a) $\nabla f = \langle 4x, 2y \rangle \quad \nabla f(2, -1) = \langle 4 \cdot 2, 2(-1) \rangle = \langle 8, -2 \rangle$

$D_{\vec{u}}(2,-1) = \nabla f(2,-1) \cdot \vec{u} = \langle 8, -2 \rangle \cdot \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle = 4\sqrt{3} - 1$

b) the direction of the steepest ascent:

$$\frac{\nabla f(2,-1)}{|\nabla f(2,-1)|} = \frac{\langle 8, -2 \rangle}{\sqrt{64+4}} = \langle \frac{8}{\sqrt{68}}, \frac{-2}{\sqrt{68}} \rangle$$

the rate of the steepest ascent:

$$|\nabla f(2,-1)| = \sqrt{64+4} = \sqrt{68}$$

DO NOT WRITE ABOVE THIS LINE!!

3. **(10 pt)** Let R be the region in the xy-plane bounded by the circle $x^2 + y^2 = 4$. Let

$$f(x, y) = \frac{1}{3}y^3 + y - x^2.$$

Find the absolute maximum and minimum values of f on R and the points at which they occur.

$\nabla f(x,y) = \langle -2x, \frac{1}{3} \cdot 3y^2 + 1 \rangle = \langle -2x, y^2+1 \rangle$

$-2x = 0, \quad y^2 + 1 = 0$

$x = 0 \quad y^2 = -1$ no solutions, so no critical points.

$g(x,y) = x^2 + y^2 - 4 = 0$

$\nabla g(x,y) = \langle 2x, 2y \rangle$

$-2x = \lambda 2x \quad \Rightarrow \quad \lambda x + x = 0$

$y^2 + 1 = \lambda 2y \quad\quad x(\lambda + 1) = 0 \quad x = 0$ or $\lambda = -1$

$x = 0$: $y^2 = 4$, $y = \pm 2$, points $(0,-2), (0,2)$

$\lambda = -1$: $y^2 + 1 = -2y$

$y^2 + 2y + 1 = 0$

$(y+1)^2 = 0 \Rightarrow y = -1$, $x^2 = 4 - (-1)^2 = 3$

$x = \pm\sqrt{3}$,

points $(-\sqrt{3}, -1)$, $(\sqrt{3}, -1)$

$f(0,-2) = \frac{1}{3}(-2)^3 - 2 = -\frac{8}{3} - 2 = -\frac{14}{3}$

$f(0,2) = \frac{1}{3} \cdot 8 + 2 = \frac{14}{3}$

$f(-\sqrt{3}, -1) = -\frac{1}{3} - 1 - 3 = -\frac{13}{3}$

$f(\sqrt{3}, -1) = -\frac{1}{3} - 1 - 3 = -\frac{13}{3}$

$\boxed{\text{abs. max } f(0,2) = \frac{14}{3} \quad \text{abs. min } f(0,-2) = -\frac{14}{3}}$

DO NOT WRITE ABOVE THIS LINE!!

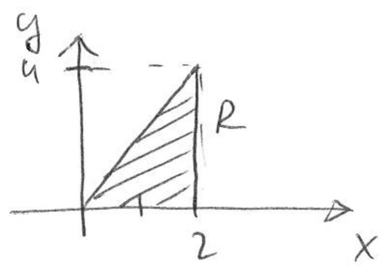
4. **(10 pt)** For the integral

$$\int_0^4 \int_{\frac{y}{2}}^2 e^{x^2}\, dx\, dy$$

(a) Sketch the region of integration.

(b) Reverse the order of integration.

(c) Evaluate the integral from (b).

a) 

$R = \{(x,y) \mid 0 \le y \le 4, \frac{y}{2} \le x \le 2\}$

$x = \frac{y}{2}$

$y = 2x$

b) $R = \{(x,y) \mid 0 \le x \le 2, 0 \le y \le 2x\}$

$$\int_0^4 \int_{\frac{y}{2}}^2 e^{x^2}\, dx\, dy = \int_0^2 \int_0^{2x} e^{x^2}\, dy\, dx = \int_0^2 y e^{x^2} \Big|_0^{2x} dx =$$

$$\int_0^2 2x e^{x^2}\, dx = \int_0^4 e^u\, du = e^u \Big|_0^4 = e^4 - 1$$

$u = x^2 \quad du = 2x\,dx$

if $x = 0$ then $u = 0$

$x = 2 \quad u = 4$

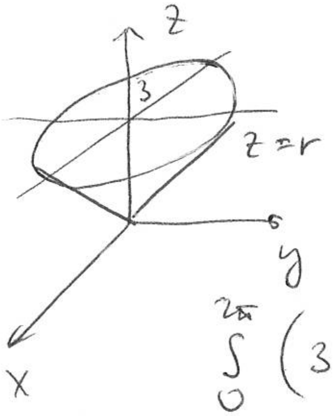
5. **(10pt)** Find the volume of the solid that is bounded by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 3$.

$3 = \sqrt{x^2+y^2}$

$x^2+y^2 = 3^2$ - circle of radius 3

Change to cylindrical coordinates: $z = \sqrt{r^2} = r$,

$D = \{(r,\theta,z) : 0 \le \theta \le 2\pi,\ 0 \le r \le 3,\ r \le z \le 3\}$



$$V = \iiint_D dV = \int_0^{2\pi}\int_0^3\int_r^3 r\,dz\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^3 rz\Big|_r^3 dr\,d\theta = \int_0^{2\pi}\int_0^3 3r - r^2\,dr\,d\theta =$$

$$\int_0^{2\pi}\left(3\frac{r^2}{2} - \frac{r^3}{3}\Big|_0^3\right)d\theta = \left(\frac{3}{2}\cdot 9 - \frac{3}{3}\cdot 9\right)\theta\Big|_0^{2\pi} = \frac{9}{2}2\pi = 9\pi$$

---

or: Change to polar coordinates:

$R = \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 3\}$, $z = \sqrt{r^2} = r$

$$V = \iint_R (3 - r)\,r\,dr\,d\theta \int_0^{2\pi}\int_0^3 (3r - r^2)\,dr\,d\theta =$$

$$\int_0^{2\pi}\left(\frac{3}{2}r^2 - \frac{r^3}{3}\right)\Big|_0^3 d\theta = \left(\frac{3}{2} - 1\right)\cdot 9\ \theta\Big|_0^{2\pi} = 9\pi$$

DO NOT WRITE ABOVE THIS LINE!!

6. **(15 pt)** Consider a vector field $\mathbf{F} = \langle 2x + y, 2y + x \rangle$.

(a) Verify that the vector field $\mathbf{F}$ is conservative by checking partial derivatives.

(b) Find a potential function for $\mathbf{F}$.

(c) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the line segment given by $\mathbf{r}(t) = \langle 2t + 1, t - 3 \rangle$ for $0 \leq t \leq 1$.

a) $\frac{\partial f}{\partial y} = 1 \quad \frac{\partial g}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x},$ conservative.

b) $\varphi = \int 2x + y \, dx = 2\frac{x^2}{2} + xy + C(y) = x^2 + xy + C(y)$

$\varphi_y = x + C'(y) = x + 2y,$ so $C'(y) = 2y$

$C(y) = \int 2y\,dy = 2\frac{y^2}{2} + K,$

take $K = 0,$

$$\boxed{\varphi = x^2 + xy + y^2}$$

c) $\vec{r}(0) = \langle 1, -3 \rangle, \quad \vec{r}(1) = \langle 3, -2 \rangle$

$\int_C \vec{F} \cdot d\vec{r} = \varphi(3,-2) - \varphi(1,-3) = (9 - 6 + 4) -$

$(1 - 3 + 9) = 7 - 7 = 0$

DO NOT WRITE ABOVE THIS LINE!!

7. **(10pt)** Use Green's theorem to compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y) = \langle x^2 y, xy^5 \rangle,$$

and C is the boundary of the square with vertices $(-1, -1)$, $(1, -1)$, $(1, 1)$ and $(-1, 1)$ oriented counterclockwise.

$$R = \{ (x, y) \mid -\cancel{0} \le x \le 1, \ -1 \le y \le 1 \}$$

$$\frac{\partial g}{\partial x} = y^5 \qquad \frac{\partial f}{\partial y} = x^2$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (y^5 - x^2)\, dA = \int_{-1}^{1} \int_{-1}^{1} (y^5 - x^2)\, dx\, dy =$$

$$\int_{-1}^{1} \left(y^5 x - \frac{x^3}{3} \right) \Big|_{-1}^{1} dy = \int_{-1}^{1} \left(y^5 - (-y^5) - \frac{1}{3}\left((+1)^3 - (-1)^3 \right) \right) dy =$$

$$\int_{-1}^{1} \left(2y^5 - \frac{2}{3} \right) dy = \left(2 \frac{y^6}{6} - \frac{2}{3} y \right) \Big|_{-1}^{1} =$$

$$\frac{1}{3} \left(1^6 - (-1)^6 \right) - \frac{2}{3} \left(1 - (-1) \right) = -\frac{4}{3}$$

DO NOT WRITE ABOVE THIS LINE!!

8. **(10 pt)**

(a) Find the gradient vector field of the function $\varphi(x, y, z) = \cos(xy + z)$.

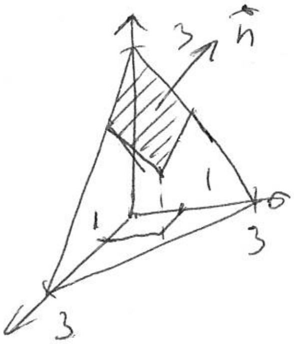
$$\nabla\varphi = \langle -y\sin(xy+z), -x\sin(xy+z), -\sin(xy+z)\rangle$$

(b) Compute the divergence and the curl of the vector field $\mathbf{F} = \langle 2z - x, x + y + z, 2y - x\rangle$.

$$\operatorname{div}\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = -1 + 1 + 0 = 0$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z-x & x+y+z & 2y-x \end{vmatrix} = \vec{i}(2-1) - \vec{j}(-1-2) + \vec{k}(1-0) = \vec{i} + 3\vec{j} + \vec{k}$$

9. **(10 pt)** Compute the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where S is the part of the plane $x + y + z = 3$ above the square $[0,1] \times [0,1]$, oriented with the upward normal, and $\mathbf{F} = \langle z, 1, x \rangle$.



$u = x, \quad v = y$

$\vec{r}(u,v) = \langle u, v, 3 - u - v \rangle$

$R = \{(u,v) \mid 0 \le u \le 1, \ 0 \le v \le 1\}$

$\vec{t}_u = \langle 1, 0, -1 \rangle, \quad \vec{t}_v = \langle 0, 1, -1 \rangle$

$\vec{t}_u \times \vec{t}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i}(-(-1)) - \vec{j}(-1) + \vec{k} \cdot 1 = \langle 1, 1, 1 \rangle$

$\iint_S \vec{F} \cdot \vec{n}\, dS = \int_0^1 \int_0^1 \langle 3 - u - v, 1, u \rangle \cdot \langle 1, 1, 1 \rangle\, dv\, du =$

$\int_0^1 \int_0^1 (3 - \cancel{u} - v + 1 + \cancel{u})\, dv\, du = \int_0^1 \int_0^1 (4 - v)\, dv\, du =$

$\int_0^1 \left(4v - \frac{v^2}{2}\right)\Big|_0^1 du = \int_0^1 \left(4 - \frac{1}{2}\right) du =$

$\frac{7}{2} u \Big|_0^1 = \frac{7}{2}$