Math 210 – Spring 2015

Name: Solutions

Midterm Exam 1
Duration: 2 hours
Total: 80 points

The following rules apply:

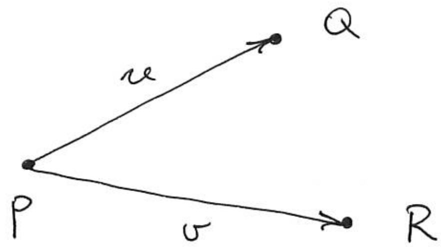
- You are expected to abide by the University's rules concerning Academic Honesty.
- You may *not* use your books, notes, or any electronic device including cell phones.
- You must show all of your work. An answer, right or wrong, without the proper justification will receive little to no credit.
- You must complete your work in the space provided.

Check next to your instructor:

Kobotis	
Lukina @ 11am	
Lukina @ 2pm	
Cole	
Levine	
Steenbergen @ noon	
Steenbergen @ 2pm	
Xie	
Shvydkoy	

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	80	

(10 pts) **1.** Find the area of the triangle with vertices at $P(3,1,0)$, $Q(3,2,0)$, $R(0,1,1)$. Determine the angle between vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.



$$\text{Area} = \frac{1}{2}|u \times v|$$

$$u = \overline{PQ} = \langle 0, 1, 0 \rangle$$

$$v = \overline{PR} = \langle -3, 0, 1 \rangle$$

$$u \times v = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{vmatrix} = \langle 1, 0, 3 \rangle$$

$$|u \times v| = \sqrt{1+9} = \sqrt{10}$$

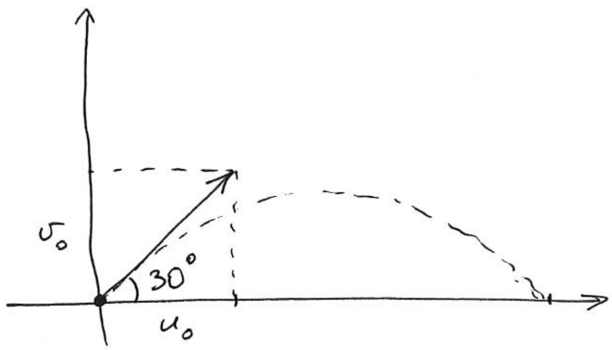
$$\boxed{\text{Area} = \frac{1}{2}\sqrt{10}}$$

To find angle we use formula

$$\cos\theta = \frac{u \cdot v}{|u|\,|v|}$$

but $u \cdot v = 0$. So, $\cos\theta = 0$, $\boxed{\theta = \frac{\pi}{2}}$

(10 pts) **2.** A football is kicked from the ground with the speed of 16 ft/s at an angle of 30 degrees. Determine the time when the ball hits the ground. Find the range and maximal height. Assume in your calculations that $g = 32ft/s^2$.



Initial velocity vector is $\langle 16\cos 30^\circ, 16\sin 30^\circ \rangle$

$= \langle 8\sqrt{3}, 8 \rangle$

The equation of trajectory is

$$r(t) = \langle 8\sqrt{3}\,t,\ 8t - 16t^2 \rangle$$

At time when the ball hits the ground

$y(t) = 0$. So, $8t - 16t^2 = 0$

$t = 0$ (initial time)

$\boxed{t = \frac{1}{2}}$

Range $= x(\frac{1}{2}) = 4\sqrt{3}$

Max height is attained when $y'(t) = 0$

$8 - 32t = 0$

$t = \frac{1}{4}$

Height $= y(\frac{1}{4}) = 2 - 1 = \boxed{1}$

(10 pts) **3.** Find an equation of the line that lies in the intersection of two planes given by

$$x - y + z = 1 \text{ and } 2y - z = 0.$$

Solution # 1

Direction vector of the line can be chosen as the vector productof the normal vectors of the planes

$$n_1 = \langle 1, -1, 1 \rangle$$
$$n_2 = \langle 0, 2, -1 \rangle$$

$$v = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 0 & 2 & -1 \end{vmatrix} = \langle -1, 1, 2 \rangle$$

To find a point pick $z = 0$, then $2y - 0 = 0$
$y = 0$

then $x - 0 + 0 = 1$
$x = 1$

So, $P(1, 0, 0)$ lies on the line.

$$\therefore \quad r(t) = \langle 1 - t, t, 2t \rangle.$$

Solution #2. Take $z = t$ to be the parameter.

Then $y = \frac{1}{2} z = \frac{1}{2} t$

$x = y - z + 1 = \frac{1}{2} t - t + 1 = -\frac{1}{2} t + 1$

So, $r(t) = \langle 1 - \frac{1}{2} t, \frac{1}{2} t, t \rangle$.

To match withe the previous, set new parameter to be $s = \frac{1}{2} t$. Then

$$r(s) = \langle 1 - s, s, 2s \rangle.$$

(10 pts) **4.** Find an equation of the plane parallel to another plane given by equation

$$3x - y + 5z = 1$$

and passing through the origin. Do not simply write the answer. Justify it.

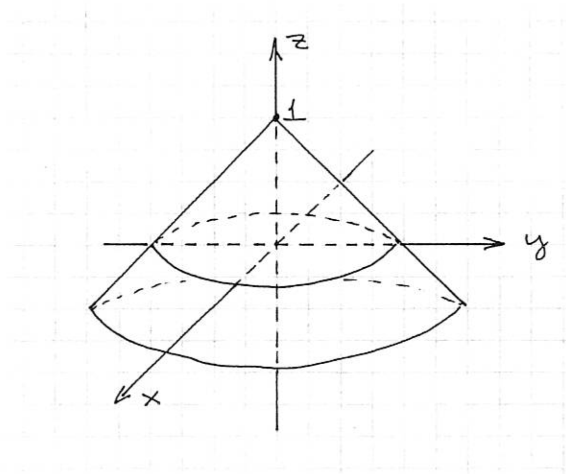
Two planes are parallel if their normal vectors are parallel. So, we can choose

$$n = \langle 3, -1, 5 \rangle .$$

Since the new plane passes through the origin $P(0,0,0)$, we write

$$3(x-0) - 1\cdot(y-0) + 5(z-0) = 0$$

$$\boxed{3x - y + 5z = 0}$$



(10 pts) **5.** Determine which one of these equations fits best for the surface pictured above. Substantiate your answer, a simple guess will not receive any credit.

(a) $x + 3y - z = 1$,

(b) $x^2 + \frac{y^2}{4} + z^2 = 1$,

(c) $z = 1 - \sqrt{x^2 + y^2}$,

(d) $z^2 = x^2 + y^2 + 1$.

Find an equation of the curve that lies in the intersection of the surface with the xy-plane.

a) is a plane, no fit.

b) is an ellipsoid, no fit

c) correct: $z = -\sqrt{x^2+y^2}$ is the lower half of the full cone $z^2 = x^2+y^2$. So, $z = 1 - \sqrt{x^2+y^2}$ is the cone lifted by 1 up as on the figure.

d) hyperboloid of two sheets.

On xy-plane $z=0$. So, $1 - \sqrt{x^2+y^2} = 0$

$1 = \sqrt{x^2+y^2}$

$\boxed{1 = x^2+y^2}$

– circle of radius 1.

(10 pts) **6.** Find the arc length of the curve parametrized by

$$\mathbf{r}(t) = \left\langle e^t, e^{-t}, \sqrt{2}t \right\rangle,$$

on the range $0 \le t \le \ln 2$.

$$L = \int_0^{\ln 2} |r'(t)|\, dt.$$

$$r'(t) = \langle e^t, -e^{-t}, \sqrt{2} \rangle$$

$$|r'(t)| = \sqrt{e^{2t} + e^{-2t} + 2}$$

$$= \sqrt{(e^t)^2 + 2e^t e^{-t} + (e^{-t})^2}$$

$$= \sqrt{(e^t + e^{-t})^2}$$

$$= e^t + e^{-t}$$

$$L = \int_0^{\ln 2} (e^t + e^{-t})\, dt = (e^t - e^{-t}) \Big|_0^{\ln 2}$$

$$= e^{\ln 2} - e^{-\ln 2} - (1 - 1)$$

$$= 2 - \frac{1}{2} = \boxed{\frac{3}{2}}$$

(10 pts) **7.** Given the function of two variables

$$f(x,y) = \frac{1}{\sqrt{xy-1}}$$

find and sketch the domain of f. Sketch the level curve $f = 1$ on the same coordinate grid.

The domain of f is detemined by the condition that $\sqrt{\cdot}$ is defined for non-negative numbers, and division by 0 is not allowed.

So,

$$xy - 1 \neq 0$$

<u>and</u>

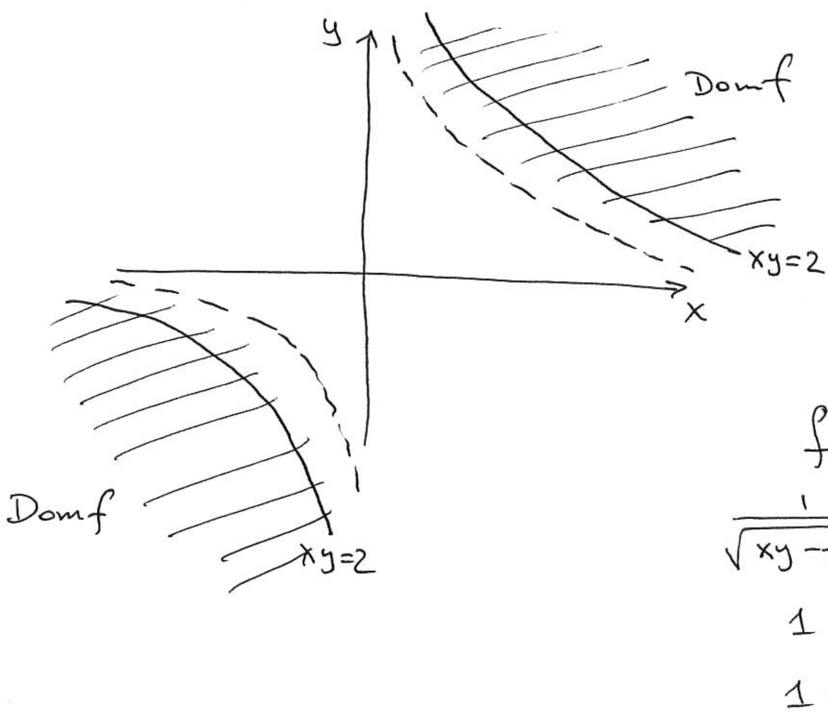
$$xy - 1 \geq 0$$

We therefore have

$$xy - 1 > 0$$
$$xy > 1$$

Equation $xy = 1$ defines a hyperbola, and it is <u>not</u> in the domain.

$xy > 1$ is the region <u>above</u> the hyperbola in the 1^{st} quadrant, and below in the 3^{rd}.



$$f = 1$$

$$\frac{1}{\sqrt{xy-1}} = 1$$

$$1 = \sqrt{xy-1}$$

$$1 = xy - 1$$

$$2 = xy$$

(10 pts) **8.** Show that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{2x^4 + y^2}$$

Let us use Two Path Test with coordinate axes:

$x = 0,\ y \to 0$: $\frac{x^4 - 4y^2}{2x^4 + y^2} = \frac{-4y^2}{y^2} = -4 = L_1$

$y = 0,\ x \to 0$: $= \frac{x^4}{2x^4} = \frac{1}{2} = L_2$

$L_1 \neq L_2$ therefore the limit DNE.